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NGUYEN TRI DAT

LIOUVILLIAN SOLUTIONS OF FIRST-ORDER ALGEBRAIC ORDINARY DIFFERENTIAL EQUATIONS

DOCTORAL DISSERTATION IN MATHEMATICS

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Declaration

This dissertation was done at the Department of Mathematics and Statistics, Quy Nhon University under the supervision of Dr. Ngo Lam Xuan Chau and Assoc. Prof. Dr. Le Cong Trinh. I hereby declare that the results presented in it are truthful and original. Most of them were published in peer-reviewed journals, others have not been published elsewhere. For using results from joint papers I have gotten permissions from my co-authors.

Binh Dinh, 2024

Nguyen Tri Dat

Abstract

Differential equations have been studied for a long time. Various exact solution methods have been proposed for special cases. The main aim of this dissertation is to develop and investigate new methods for determining liouvillian solutions of first-order algebraic ordinary differential equations (AODEs). For this purpose, the differential problem is transformed into an algebraic geometric one by considering the differential equation to be an algebraic equation. Such an equation defines an algebraic curve and therefore, tools from algebraic geometry can be applied. In particular, parametrizations of algebraic curves and algebraic function fields are intrinsically used to solve the problem and prove properties of the obtained solutions.

A first idea for determining rational liouvillian solutions of first-order autonomous AODEs is presented. This approach is a generalization of a well-known algorithm for finding rational solutions. It admits an extension to the computation of the liouvillian solutions which is obtained by considering the wider differential fields.

A second focus lies on the extension of the first idea to the problem of finding liouvillian solutions of first-order autonomous AODEs of genus zero. In this situation, the theory of associated fields of algebraic functions is applied to prove a liouvillian solution (if there exists) must be a rational liouvillian solution. This leads to a classification of the liouvillian solutions respect to algebraic and transcendental cases.

Last focus studies liouvillian solutions of first-order AODEs. If an AODE is of genus zero, we prove that its liouvillian solutions can be found via first-order quasi-linear ODEs by means of associated fields of algebraic functions and optimal rational parametrizations. This method inherits the approach of existing algorithms for finding rational general solutions. Finally, we present an approach for solving first-order AODEs of positive genera by means of power transformations.

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Nguyen Tri Dat

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List of algorithms

Table of notations

$\mathbf{k}$	:	A differential field of characteristic zero
$\mathbb{K}$	:	The field of constants of a differential field $\mathbf{k}$
E	:	The differential extension field of $\mathbf{k}$
$\mathbb{K}(x)$	:	The differential field of rational functions in x with constants $\mathbb{K}$
$\overline{\mathbb{Q}}$	:	The algebraic closure field of rational numbers
$\mathbb{C}$	:	The field of complex numbers
K	:	An algebraic closed field of characteristic zero
$\mathbb{A}^2(K)$	:	The affine plane over K
$\mathbb{P}^2(K)$	:	The projective plane over K
$K[t]; K[x,y]$	:	The polynomial ring of one; two variables over K
$\mathbb{K}(x)\{y\}$	:	The ring of differential polynomials in y over $\mathbb{K}$
F	:	A polynomial in $K[x,y]$ or differential polynomial in $\mathbb{K}(x)\{y\}$
Σ	:	A prime differential ideal
Σ_i	:	Essential prime differential ideals
$\{F\}$	:	The radical differential ideal generated by the differential polynomial F
S_F	:	The separant of the differential polynomial F
$\deg_x G$	:	The degree of G with respect to x
$\mathcal{V}(F)$	:	The zero set of $F \in K[x,y]$ in $\mathbb{A}^2(K)$
$\mathcal{C} = \mathcal{V}(F)$	:	The affine algebraic curve defined by F
$\hat{F}$	:	The homogenization of $F \in K[x,y]$
$\Gamma = \mathcal{V}(\hat{F})$	:	The projective closure of the affine algebraic curve $\mathcal{C}$
$g(\Gamma)$	:	The genus of the projective curve Γ
$R(\mathbb{P}^2)$	:	The field of rational functions on $\mathbb{P}^2(K)$
$R(\mathbb{A}^2)$	:	The field of rational functions on $\mathbb{A}^2(K)$
$R(\hat{F})$	:	The field of rational functions on $\hat{F}$
$E; L; K(\eta, \xi)$	:	The field of algebraic functions of one variable
$K(t)$	:	The field of rational functions of one variable
$K((t))$	:	The power series field with the parameter t
$\mathrm{res}_t(A,B)$	:	The Sylvester resultant respect to $A, B \in K[t]\backslash\{0\}$
L_P	:	The power series field on P

$\exp x$	:	The exponent of x
$\log x$	:	The inverse function of $\exp x$
$\mathfrak{o}$	:	The valuation ring
$\mathfrak{p}$	:	The place (of a valuation ring)
$\Sigma_{\mathfrak{p}}$	:	The residue field at $\mathfrak{p}$
$v_{\mathfrak{p}}(x)$	:	The order of x at $\mathfrak{p}$
$x(\mathfrak{p})$	:	The value of x at $\mathfrak{p}$
$\mathfrak{a}$	:	The divisor over K
$\deg(\mathfrak{a})$	:	The degree of $\mathfrak{a}$
$\mathfrak{L}(\mathfrak{a})$	:	The vector space over K of a divisor $\mathfrak{a}$
$l(\mathfrak{a})$	:	The dimension of the vector space $\mathfrak{L}(\mathfrak{a})$ over K
g_L	:	The genus of the function field L

Introduction

A *differential equation* (DE) is an equation that includes one or more unknown functions and their derivatives. The history of DEs can be traced back to the invention of calculus by Newton (in physics) and Leibniz (in pure mathematics) around 1660s–1670s. In application, the functions generally represent physical quantities, the derivatives represent their rates of change, and the DE defines a relationship between the two. Hence, these DEs play a prominent role in many disciplines including physics, engineering, economics, and biology. If a DE contains an unknown function and its derivatives which depend on an independent variable x then it is called an *ordinary differential equation* (ODE). A DE is called *linear* if the relationship of the unknown function and its derivatives is linear; otherwise, it is called *nonlinear*. Such DEs can exhibit very complicated behavior over extended time intervals, characteristic of chaos. Unfortunately, there are very few methods of solving nonlinear DEs exactly. Most ODEs encountered in physics are linear; hence, there are many ways for solving them. An idea of transforming nonlinear DEs into linear DEs and then solve the last ones may be a reasonable candidate. However, it works for only some cases. Therefore, studying independently the solutions of nonlinear DEs is necessary, and it also contains a lot of challenges. In this dissertation, we study liouvillian general solutions of first-order *algebraic ordinary differential equations* (AODEs) which is a fundamental problem in the theory of non-linear algebraic DEs.

A first-order AODE is a DE of the form $F(y, y') = 0$, where F is an irreducible polynomial in two variables with coefficients in $\mathbb{K}(x)$, $\mathbb{K}$ is an algebraically closed field of characteristic zero. Solving an AODE is a problem of determining differentiable functions $y = y(x)$ satisfying $F(y(x), y'(x)) = 0$. If $y(x)$ belongs to $\mathbb{K}(x)$ (resp. an algebraic extension field of $\mathbb{K}(x)$), then it is called a *rational solution* (resp. an *algebraic solution*). If such a solution $y(x)$ belongs to a liouvillian extension of $\mathbb{K}(x)$, then it is called a *liouvillian solution*. A solution may contain an arbitrary constant. In this case, such a solution is called a *general solution*. For example, $y(x) = \exp(x^2 + c)$ is a *liouvillian general solution* of the first-order AODE $y' - 2xy = 0$.

First-order AODEs have been studied a lot and there are many solution methods for their special classes. The study of these AODEs can be dated back to the works of Fuchs [16] (1884). In [20] (1926), Ince presented an overall picture of ODEs. In [30,31] (1970s), Matsuda classified differential function fields having no movable critical points up to isomorphism of differential fields. By focusing on particular solutions, in [29] (1913), Malmquist studied the class of first-order AODEs having transcendental meromorphic solutions, and Eremenko revisited later in [10] (1982). Applied Matsuda's theory, Eremenko in [11] (1998) gave a theoretical consideration on a degree bound for rational solutions which sheds light on the issue of finding the solution's explicit form.

Finding the closed form solution of an ODE can be traced back to the works of Liouville (1830s) for the simplest ODE $y' = \alpha$, where $\alpha \in \mathbf{k}$ and $\mathbf{k}$ is a differential field of characteristic zero. If such an equation has a solution in some elementary differential extension field E of $\mathbf{k}$ having the same subfield of constants $\mathbb{K}$, then there exist constants $c_1, c_2, \ldots, c_n \in \overline{\mathbb{K}}$, elements $u_1, u_2, \ldots, u_n \in \overline{\mathbb{K}}\mathbf{k}$ and $v \in \mathbf{k}$ such that

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'.$$

In [44] (1968), Rosenlicht showed how Liouville theorem can be handled algebraically. For the algorithm consideration of such ODE, the pioneer work is due to Risch. In [41, 42] (1960s), Risch described a method to determine an *elementary integral* $\int u$ where u is an *elementary function*. To extend Risch's method, in [51, 52] (1970s), Singer studied *elementary solutions* of first-order AODEs. As a special result, there are necessary and sufficient conditions for the ODE $y' = R(y) \in \mathbb{C}(y)$ having an elementary solution. In [56] (2017), Srinivasan generalized this result to the case of liouvillian solutions with the same conditions. In [25] (1986), Kovacic presented an effective method to find liouvillian solutions of second order linear homogeneous ODEs. This work contains an algorithm for finding rational general solutions of a Riccati equation which is applicable to the works of Chen and Ma [7] (2005) and Vo et al. [57] (2018) for determining rational general solutions of first-order parametrizable AODEs. In [19] (1996), Hubert studied implicit general solutions of $F(y, y') = 0$ by computing Gröbner bases. In [40] (1983), Prelle and Singer studied *elementary first integrals* $I(x, y)$ of the following system of ODEs

$$\frac{dx}{dz} = P(x, y); \ \frac{dy}{dz} = Q(x, y), \ \text{where } P(x, y), Q(x, y) \in \mathbb{C}[x, y].$$

Such a first integral induces a general solution $I(x, y) = c$ of the ODE $y' = \dfrac{Q(x, y)}{P(x, y)}$. In [55] (1992), Singer revisited this problem for finding *liouvillian first integrals* $I(x, y)$. In computational aspects, recently, Duarte and Da Mota, [9] (2021), presented an efficient method for computing liouvillian first integrals.

The starting point for the algebro-geometric method was algorithms introduced by the works of Feng and Gao in [14, 15] (2000s). These algorithms decide whether or not an autonomous first-order AODE, $F(y, y') = 0$, has a rational general solution and compute it if there is any. The key point is that a rational solution of such an AODE induces a proper rational parametrization of the corresponding algebraic curve, from that, we find a reparametrization such that the second component is the derivative of the first one. The existence of a proper parametrization can be decided by the works of Sendra and Winkler [49] (2001). From that, a rational general solution can be deduced.

Using the ideas of Feng and Gao in [14, 15], several generalizations have been investigated since then. There are (not exhausted) notable works. In [7], by means of rational parametrizations, Chen and Ma reduced the problem of determining rational general solutions of a first-order parametrizable AODE to the case of solving a Riccati equation. In here, the method in [25] for finding rational general solutions (of a Riccati equation) can be applied. This work is not complete due to the rational forms of the rational parametrizations over a rational function field are required. In [33, 34] (2010s), Ngo and Winkler introduced a method based on parametrizations of surfaces for finding rational general solutions of such parametrizable AODEs. In [57], by determining an optimal parametrization of an algebraic curve over a rational function field, Vo et al. overcame the missing steps of [7] and obtained a decision algorithm of finding strong rational general solutions of first-order AODEs. A summarization and more aspects of the algebro-geometric method can be found in Sebastian et al. [12] (2023).

In this dissertation, we inherit and extend the works by Feng and Gao [14, 15], Srinivasan [56], and Vo et al. [57] for determining liouvillian solutions of first-order AODEs. In particular, the dissertation contributes the following results.

- Define rational liouvillian solutions (see Definition 2.2.3) and give an algorithm (see Algorithm `RatLiouSol` in Section 2.4) for finding such rational liouvillian solutions of first-order autonomous AODEs.
- Show that a liouvillian solution (which includes the class of algebraic solutions) of a first-order autonomous AODE of genus zero must be a rational liouvillian solution (see Lemma 3.2.2) and give an algorithm (see Algorithm `LiouSolAut` in Section 3.3) for finding and classifying such a liouvillian solution in algebraic and transcendental cases.
- Propose an algorithm (see Algorithm `LiouSol` in Section 4.1.2) for determining liouvillian solutions of first-order AODEs of genus zero (included autonomous and non-autonomous cases).

- Define power transformations (see Definition 4.2.1) and give an algorithm (see Algorithm `RedPol` in Section 4.2.2) to obtain reduced forms of first-order AODEs. This result leads to a method for finding liouvillian solutions of first-order AODEs of positive genera in case their reduced forms are of genus zero (see Section 4.2.3).
- Transform the problem of solving first-order AODEs with liouvillian coefficients into the case of solving an AODE (4.1) by means of change of variables (see Section 4.4).

In this dissertation, we summarize our works in the last three years and give short description of our future research. The dissertation is organized as follows.

Chapter 1 presents basic materials in differential algebra and algebraic geometry. It also contains the main tools using regularly in the dissertation.

In Chapter 2, we define rational liouvillian solutions of first-order autonomous AODEs. Using the properties of rational parametrizations of algebraic curves, we give necessary and sufficient conditions for a first-order autonomous AODE to have rational liouvillian solutions. Based on this, we present an algorithm for determining rational liouvillian solutions of first-order autonomous AODEs.

In Chapter 3, we apply the theory of fields of algebraic functions of one variable to show that a liouvillian solutions of a first-order autonomous AODE of genus zero, if there exists, must be a rational liouvillian solution. By using Sylvester resultant, the forms of the liouvillian solutions can be described in algebraic relations. These results lead to an algorithm for determining the existence of such liouvillian solutions.

In Chapter 4, we study liouvillian solutions of first-order AODEs of genus zero via their associated ODEs by means of rational parametrizations. Using the theory of fields of algebraic functions of one variable, we prove that the property of having a liouvillian general solution of the two above differential equations are the same. This result covers the autonomous case considered in Chapter 3. If first-order AODEs are of positive genera, then there is an approach for solving them. First, we give an algorithm to compute a reduced form of a given AODE by means of power transformations. From that, we present a method for finding liouvillian solutions of certain first-order AODEs of positive genera whose reduced forms are of genus zero. In final consideration, we study the problem of solving first-order AODEs with coefficients in a liouvillian extension of $\mathbb{C}(x)$ by means of change of variables.

Chapter 1

Preliminaries

In this chapter, we briefly recall some basic notions and definitions in differential algebra and algebraic geometry which are tailored for the dissertation. The chapter is organized as follows. Section 1.1 introduces algebraic ordinary differential equations and their general solutions. Section 1.2 gives the definitions of plane algebraic curves and the computation of their genera. Section 1.3 presents general concepts of fields of algebraic functions of one variable. Section 1.4 shows the models of fields of algebraic functions related to projective algebraic curves. Section 1.5 prepares the main tools which are using regularly in this dissertation: associated algebraic function fields and rational parametrizations. More details of differential algebra and algebraic geometry can be found in the standard textbooks such as [4,24,43] and [8,27,50,59], respectively.

1.1 Differential algebra

Definition 1.1.1. Let $\mathbf{k}$ be a field of characteristic zero. A *derivation* of the field $\mathbf{k}$, denote by $'$, is an operation of $\mathbf{k}$ that satisfies the two following items:

1. $(a+b)' = a' + b'$
2. $(ab)' = a'b + ab'$

for every $a, b \in \mathbf{k}$. A field $\mathbf{k}$ equipped with a derivation $'$ is called a *differential field*. An element $a \in \mathbf{k}$ is called a *constant* if $a' = 0$.

Definition 1.1.2. A field extension E of $\mathbf{k}$ is called a *differential field extension* of $\mathbf{k}$ if and only if the derivation of E restricted to $\mathbf{k}$ coincides with the derivation of $\mathbf{k}$.

Remark 1.1.3. Every field $\mathbb{K}$ can be seen as a differential field of constants with the trivial derivation which maps all elements of the field to zero. Let $\mathbb{K}(x)$ be the field of rational functions in x with coefficients in $\mathbb{K}$. The trivial derivation on $\mathbb{K}$ can be extended to a derivation $' = \dfrac{d}{dx}$ on $\mathbb{K}(x)$ in such a way that $x' = \dfrac{d}{dx}(x) = 1$.

In general, we consider the field of rational functions

$$\mathbb{K}(x_1, x_2, \ldots, x_n)$$

for some algebraically closed field $\mathbb{K}$ of characteristic zero. By $\dfrac{\partial}{\partial x_i}$ we denote the usual derivative by x_i. In the case of $n = 1$, we also write x for x_1 and $'$ or $\dfrac{d}{dx}$ for $\dfrac{\partial}{\partial x_1}$. Then such field of rational functions $\mathbb{K}(x_1, x_2, \ldots, x_n)$ together with the derivations is a differential field. The *ring of differential polynomials* is denoted by

$$R := \mathbb{K}(x_1, x_2, \ldots, x_n)\{y\}$$

which consists of all polynomials in y and its derivatives. For higher-order derivatives we use

$$y^{(1)} = y' \text{ and } \text{ recursively } y^{(k)} = (y^{(k-1)})'.$$

Definition 1.1.4. An ideal Σ in R is called a *differential ideal* if it is closed under the derivation, i.e. $\Sigma' \subseteq \Sigma$. In addition, if Σ is prime (resp. radical) ideal of R, it is called a prime (resp. radical) differential ideal.

This dissertation mainly focuses on the ring of differential polynomials $\mathbb{K}(x)\{y\}$, and the problem of finding liouvillian solutions of a polynomial F of order one in it. From now on we denote by $\mathbb{K}$ an algebraically closed field of characteristic zero with the trivial derivation. In practice, we might choose $\mathbb{K} = \mathbb{C}$, the field of complex numbers.

Definition 1.1.5. An algebraic ordinary differential equation (AODE) is defined by an irreducible differential polynomial $F \in \mathbb{K}(x)\{y\}$ whose coefficients in $\mathbb{K}(x)$. An AODE is called *autonomous* if $F \in \mathbb{K}\{y\}$, that is if the coefficients of F do not depend on the variables of differentiation x. An AODE is called non-autonomous if it is not necessarily autonomous.

Definition 1.1.6. Let Σ be a differential ideal in $\mathbb{K}(x)\{y\}$ and let E be a differential field extension of $\mathbb{K}$. We call $\xi = (\xi_1, \xi_2, \ldots, \xi_k) \in E^k$ a *zero* of Σ if it vanishes all of elements of Σ.

Definition 1.1.7. Let Σ be a prime differential ideal in $\mathbb{K}(x)\{y\}$ and ξ be a zero of Σ. Then we call ξ a *generic zero* of Σ if for any differential polynomial $P \in \mathbb{K}(x)\{y\}$, if $P(\xi) = 0$ then P must be in Σ.

Remark 1.1.8. It is well-known that every prime differential ideal has a generic zero.

Let F be an irreducible differential polynomial of order n, the *separant* of F is denoted by

$$S_F = \frac{\partial F}{\partial y^{(n)}}.$$

Due to Ritt [43, Chapter II], the *radical differential ideal* $\{F\}$ generated by F, can be decomposed by the $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ as follows

$$\{F\} = \Sigma_1 \cap \Sigma_2 \cap \ldots \cap \Sigma_k.$$

Note that, a prime divisor of an ideal is called *essential* if it does not contain any other prime divisor. We can rewrite the above representation by

$$\{F\} = \Sigma_F \cap \{F, S_F\},$$

where

$$\Sigma_1 = \Sigma_F = \{F\} : \langle S_F \rangle = \{P \in \mathbb{K}(x)\{y\} \mid S_F P \in \{F\}\}$$

is a prime differential ideal which does not vanish S_F, and

$$\Sigma_2 \cap \ldots \cap \Sigma_k = \{F, S_F\}.$$

This Σ_F is called the *general component*, and the other part $\{F, S_F\}$ is called the *singular component*.

Definition 1.1.9. A zero of $\{F\}$ is called a solution of the AODE $F = 0$. A solution is called *non-singular* if it fails to annul the separant. Otherwise, it is called *singular*.

Definition 1.1.10. A generic zero of Σ_F is called a *general solution* of the AODE $F = 0$. A general solution depends on some transcendental constant.

Remark 1.1.11. The non-singular zeros are all contained in the general component. Every non-singular solutions can be expressed by a certain evaluation of the constants in the general solution. No choice of evaluating the constant yields a singular solution.

Let E be a differential field extension of $\mathbf{k}$ and let $'$ denote the derivation on it.

Definition 1.1.12. ([4, Definition 5.1.1]) $t \in E$ is *primitive* over $\mathbf{k}$ if $t' \in \mathbf{k}$. $t \in E \backslash 0$ is *hyperexponential* over $\mathbf{k}$ if $t'/t \in \mathbf{k}$. $t \in E$ is *liouvillian* over $\mathbf{k}$ if t is either algebraic, or primitive, or hyperexponential over $\mathbf{k}$. E is a *liouvillian extension* of $\mathbf{k}$ if

$$E = \mathbf{k}(t_1, t_2, \ldots, t_n),$$

and there is a tower of differential fields

$$\mathbf{k} = \mathbf{k}_0 \subseteq \mathbf{k}_1 \subseteq \cdots \subseteq \mathbf{k}_n = E$$

such that for each $i \in \{1, \ldots, n\}$, $\mathbf{k}_i = \mathbf{k}_{i-1}(t_i)$ and t_i is liouvillian over $\mathbf{k}_{i-1}$.

Definition 1.1.13. ([4, Definition 5.1.3]) $t \in E$ is a *logarithm* over $\mathbf{k}$ if $t' = b'/b$ for some $b \in \mathbf{k}\backslash 0$. $t \in E\backslash 0$ is *exponential* over $\mathbf{k}$ if $t'/t = b'$ for some $b \in \mathbf{k}$. $t \in E$ is *elementary* over $\mathbf{k}$ if t is either algebraic, or exponential, or a logarithm over $\mathbf{k}$.

Definition 1.1.14. E is an *elementary extension* of $\mathbf{k}$ if

$$E = \mathbf{k}(t_1, t_2, \ldots, t_n),$$

and there is a tower of differential fields

$$\mathbf{k} = \mathbf{k}_0 \subseteq \mathbf{k}_1 \subseteq \cdots \subseteq \mathbf{k}_n = E$$

such that for each $i \in \{1, \ldots, n\}$, $\mathbf{k}_i = \mathbf{k}_{i-1}(t_i)$ and t_i is elementary over $\mathbf{k}_{i-1}$.

We write $t = \log b$ when t is a logarithm over $\mathbf{k}$ such that $t' = b'/b$, and $t = \exp b$ when t is exponential over $\mathbf{k}$ such that $t'/t = b'$.

Remark 1.1.15. Since logarithms are also primitive and exponential elements are also hyperexponential, then an elementary extension of $\mathbb{K}(x)$ is also a liouvillian extension of $\mathbb{K}$. However, the inverse is not true, i.e. a liouvillian extension of $\mathbb{K}$ is not necessarily an elementary extension of $\mathbb{K}(x)$, for instance, see Example 1.1.19.

Definition 1.1.16. Consider the algebraic ordinary differential equation

$$F(y, y', \ldots, y^{(n)}) = 0.$$

Let ξ be a solution which is contained in a differential field E extended from $\mathbb{K}(x)$. We denote by $\mathbb{K}$ the field of constants of E.

1. ξ is called an *algebraic solution* if there is a non-zero polynomial $G \in \mathbb{K}[x, y]$ such that $G(x, \xi) = 0$. In this case, G is called an *annihilating polynomial* of ξ.

2. If $\deg_y G = 1$, then ξ is called a *rational solution*.

3. ξ is called a *liouvillian solution* (resp. *elementary solution*) if it belongs to some liouvillian (resp. elementary) extension of $\mathbb{K}(x)$.

4. ξ is called a liouvillian (resp. elementary, algebraic, rational) general solution if it is a general solution and liouvillian (resp. elementary, algebraic, rational).

Remark 1.1.17. By [1, Lemma 2.4], if $G(x, y)$ is an irreducible annihilating polynomial of ξ, then all roots $y = y(x)$ of the algebraic equation $G(x, y) = 0$ are solutions of the differential equation. Therefore, by abuse of notation, such algebraic equation G is sometimes called an algebraic solution.

This dissertation aims to study liouvillian general solutions of a first-order AODE

$$F(y, y') = 0, \tag{1.1}$$

where F is an irreducible polynomial in $\mathbb{C}(x)[y, w]$. A liouvillian general solution of an AODE (1.1) is a liouvillian solution which fails to annul the separant $S_F = \dfrac{\partial F}{\partial y'}$.

Example 1.1.18. [19, Example 2.4] Consider the first-order autonomous AODE

$$F(y, y') = y'^2 - 4y = 0.$$

We obtain $y = (x+a)^2$ is a general solution, where $a \in \mathbb{C}$ is an arbitrary constant. The separant of F is $S_F = 2y'$ which has a solution $\overline{y} = c \in \mathbb{C}$. If we choose $c = 0$, then the solution of separant is also the solution of F. Note that $\overline{y}$ can not be obtained from the general solution $y = (x+a)^2$ by specializing a. As seen in above introduction, $\overline{y} = 0$ is a singular solution.

Example 1.1.19 shows that a liouvillian solution over $\mathbb{C}$ of an AODE (1.1) may be not necessarily an elementary solution over $\mathbb{C}(x)$.

Example 1.1.19. ([4, Example 6.3.2]) Consider first-order AODE

$$F(y, y') = y' - 2xy - 1 = 0.$$

Since $S_F = 1$, then F has no singular solution. The above AODE has a general solution

$$y(x) = \exp x^2 \left(\int \exp(-x^2)dx + c \right)$$

which is a liouvillian solution over $\mathbb{C}$ since it belongs to a liouvillian extension

$$E = \mathbb{C}\left(x, \exp(x^2), \int \exp(-x^2)dx + c\right) \supset \mathbb{C}\left(x, \exp(x^2)\right) \supset \mathbb{C}(x).$$

Since the integration

$$\int \exp(-x^2)dx$$

is not an elementary function (see [44] or [4, Chapter 6]), then the liouvillian extension E is not an elementary extension over $\mathbb{C}(x)$. Hence, $y(x)$ is not an elementary solution.

Remark 1.1.20. Since the class of liouvillian solutions includes elementary solutions, then if an AODE (1.1) has no liouvillian solutions then it has no other solutions in Definition 1.1.16. By [59, Chapter 3], an AODE (1.1) has only finite singular solutions. Therefore, the liouvillian solutions obtained by the dissertation's method are always the general ones since they contain an arbitrary $c \in \mathbb{C}$ which generates infinitely many solutions. For a glance but reasonable enough information of such general solutions, see [12, Section 2]. For fully details, we refer to [43, Chapter II].

1.2 Plane algebraic curves

Let K be an algebraically closed field of characteristic zero. Let us denote $\mathbb{A}^2(K) := K^2$ the affine plane over K and $K[X,Y]$ the polynomial algebra in the variables X and Y over K. For $F \in K[X,Y]$, then the *zero set* of F is

$$\mathcal{V}(F) := \{(x,y) \in \mathbb{A}^2(K) \mid F(x,y) = 0\}.$$

Definition 1.2.1. A subset $\mathcal{C} \subset \mathbb{A}^2(K)$ is called an *affine algebraic curve* (a curve, for briefly) if there is a non-constant irreducible polynomial $F \in K[X,Y]$ such that $\mathcal{C} = \mathcal{V}(F)$. Such F is called the *defining polynomial* of $\mathcal{C}$. By abuse of notation, we sometime call $F(x,y) = 0$ an affine algebraic curve.

The *projective plane* $\mathbb{P}^2(K)$ over an algebraically closed field K is the set of all lines in K^3 through the origin. The points $P \in \mathbb{P}^2(K)$ will therefore be given by triples $\langle x_0, x_1, x_2\rangle$ with $(x_0,x_1,x_2) \in K^3\backslash(0,0,0)$, where $\langle x_0,x_1,x_2\rangle = \langle y_0,y_1,y_2\rangle$ if $(x_0,x_1,x_2) = \lambda(y_0,y_1,y_2)$ for some $\lambda \in K\backslash\{0\}$. The triple (x_0,x_1,x_2) is called a *system of homogeneous coordinates* for $P = \langle x_0,x_1,x_2\rangle$. Let $K[X_0,X_1,X_2]$ be the polynomial algebra over K in the variables X_0,X_1,X_2. If $\hat{F} \in K[X_0,X_1,X_2]$ is a homogeneous polynomial and $P = \langle x_0,x_1,x_2\rangle$ is a point of $\mathbb{P}^2(K)$, we then call P a *zero* of $\hat{F}$ if $F(x_0,x_1,x_2) = 0$. If $\deg_{\hat{F}} = d$, then

$$\hat{F}(\lambda X_0, \lambda X_1, \lambda X_2) = \lambda^d \hat{F}(X_0,X_1,X_2) \text{ for any } \lambda \in K,$$

and therefore the condition $\hat{F}(x_0,x_1,x_2) = 0$ does not depend on the particular choice of homogeneous coordinates for P. Hence, we can write $\hat{F}(P) = 0$. The set

$$\mathcal{V}(\hat{F}) := \{P \in \mathbb{P}^2(K) \mid \hat{F}(P) = 0\}$$

is called the *zero set* of $\hat{F}$ in $\mathbb{P}^2(K)$.

Definition 1.2.2. A subset $\Gamma \subset \mathbb{P}^2(K)$ is called a *projective algebraic curve* if there is a non-constant irreducible homogeneous polynomial $\hat{F} \in K[X_0,X_1,X_2]$ such that $\Gamma = \mathcal{V}(\hat{F})$. Such $\hat{F}$ is called the *defining homogeneous polynomial* of Γ.

We have an injection given by

$$i : \mathbb{A}^2(K) \to \mathbb{P}^2(K),\ i(x,y) = \langle 1,x,y\rangle\,,$$

from the affine plane to the projective plane. We identify $\mathbb{A}^2(K)$ with its image under i. Then $\mathbb{A}^2(K)$ is the complement of the line $X_0 = 0$ in $\mathbb{P}^2(K)$. This line is called the *line at infinity* of $\mathbb{P}^2(K)$; the points of this line are called *points at infinity*, and the points of $\mathbb{A}^2(K)$ are called *points at finite distance.* For $P = \langle 1,x,y\rangle$, we call (x,y) the *affine coordinates* of P.

Definition 1.2.3. Given an irreducible polynomial $F \in K[X, Y]$ with $\deg_F = d > 0$, we can define by

$$\hat{F}(X_0, X_1, X_2) := X_0^d F\left(\frac{X_1}{X_0}, \frac{X_2}{X_0}\right)$$

a homogeneous polynomial $\hat{F} \in K[X_0, X_1, X_2]$ with $\deg_{\hat{F}} = d$. Such $\hat{F}$ is called the *homogenization* of F. Moreover, let $\mathcal{C} \subset \mathbb{A}^2(K)$ be an affine algebraic curve defining by F, and let $\hat{F}$ be its homogenization. The projective algebraic curve $\Gamma = \mathcal{V}(\hat{F})$ is called the *projective closure* of $\mathcal{C}$.

We can give the following description for $\hat{F}$. If F is of degree $d > 0$ and

$$F = F_0 + F_1 + \cdots + F_d$$

where F_i is homogeneous of degree i, then

$$\hat{F} = X_0^d f_0(X_1, X_2) + X_0^{d-1} f_1(X_1, X_2) + \cdots + X_0 F_{d-1}(X_1, X_2) + F_d(X_1, X_2).$$

Definition 1.2.4. For a homogeneous polynomial $\hat{F} \in K[X_0, X_1, X_2]$, we call the polynomial $F \in K[X, Y]$ given by $F(X, Y) = \hat{F}(1, X, Y)$ the *dehomogenization* of $\hat{F}$.

Definition 1.2.5. Let $\Gamma \subset \mathbb{P}^2(K)$ be a projective algebraic curve defined by $\hat{F} \in K[X_0, X_1, X_2]$. Let $P = \langle x_0, x_1, x_2 \rangle$ be a point on Γ. Then P is called a *simple point* on Γ if and only if there is at least a $\lambda \in \{0, 1, 2\}$ such that

$$\frac{\partial \hat{F}}{\partial X_\lambda}(P) \neq 0.$$

If P is not simple, then P is called a *multiple point* or *singularity* on Γ. Let m be a number that for all $i + j + k < m$ the partial derivatives

$$\frac{\partial^{i+j+k} \hat{F}}{\partial {X_0}^i \partial {X_1}^j \partial {X_2}^k}(P)$$

vanish at P, but at least one of the partial derivatives of order m does not vanish at P. Then m is called a *multiplicity* of P on Γ. An algebraic curve Γ is called a *smooth* (or *non-singular*) curve if it has no singularity.

The genus of Γ can be computed by the following formula

$$g = \frac{(d-1)(d-2)}{2} - \sum_{P \in S} \delta_P, \tag{1.2}$$

where d is the degree of Γ, δ_P is the delta invariant of the singular point P, and S is the singular locus of the algebraic curve. If the singular point P is ordinary (i.e. the tangents at this point are distinct, see [59, Chapter III]), then

$$\delta_P = \frac{m_P(m_P - 1)}{2},$$

where m_P is the multiplicity of the singular point P.

Definition 1.2.6. The *genus* of an affine algebraic curve $\mathcal{C} = \mathcal{V}(F)$ is defined by the genus of its projective closure $\Gamma = \mathcal{V}(\hat{F})$.

Remark 1.2.7. Formula (1.2) stands for the general case of singularities. If Γ has non-ordinary singularities, by certain quadratic transformations (see [59, Chapter III]), one can birationally transform Γ into a curve with only ordinary singularities, in which the genus formula in [50, Theorem 3.6] can be used. For more details of the genus's computation, we refer the readers to [50, Chapter 3]. By using software, MAPLE may help us to find the singularities as well as the genus of an irreducible algebraic curve.

```
> with(algcurves):
> F:=F(x,y);
> degree(F,{x,y})
> singularities(F,x,y) # find the singularities of F
> genus(F,x,y); # compute genus
> with(algcurves):
> G:=x^4+x^2y^2-y^2; # (see [59, page 83])
> degree(G,{x,y})
> 4
> singularities(G,x,y) # find the singularities of G
> [[0, 0, 1], 2, 2, 2],[[0, 1, 0], 2, 1, 2]
> # the first one is non-ordinary
> g=(4-1)*(4-2)/2-(2+1)=0
> genus(G,x,y); # compute genus
> 0
> H:=y^2-x^3-1;
> degree(H,{x,y})
> 3
> singularities(H,x,y) # find the singularities of H
> an empty set # H is a non-singular curve
> g=(3-1)*(3-2)/2=1
> genus(G,x,y); # compute genus
> 1
```

1.3 Fields of algebraic functions of one variable

This section presents the concepts of fields of algebraic functions of one variable. More details can be found in standard textbooks such as [8, 27].

Definition 1.3.1. [8, Page 2] Let K be an algebraic closed field of characteristic zero. A field $L \supset K$ is called *a field of algebraic functions of one variable* over K if it satisfies the following condition: L contains an element x which is transcendental over K, and L is algebraic of finite degree over $K(x)$.

If there is no confusion, we may call such a field of algebraic functions of one variable L by an *algebraic function field* or briefly a *function field*, see [27].

Definition 1.3.2. Let L be a field and K a subfield of L. A subring $K \subset \mathfrak{o} \subsetneq L$ is called a *valuation ring* over K if it has the property that for any $x \in L$ we have $x \in \mathfrak{o}$ or $x^{-1} \in \mathfrak{o}$. It K is determined, we call $\mathfrak{o}$ a *valuation ring*.

Lemma 1.3.3. $\mathfrak{o}$ *is a local ring.*

Definition 1.3.4. Let L be an algebraic function field over K. A subset $\mathfrak{p}$ of L is called a *place* in L if it is the ideal of non-units of some valuation ring $\mathfrak{o}$ (over K) of L.

From [8, page 2], the valuation ring $\mathfrak{o}$ is uniquely determined when $\mathfrak{p}$ is given. Since every element of $\mathfrak{o}$ not in $\mathfrak{p}$ is a unit in $\mathfrak{o}$, we see immediately that the *residue ring* $\Sigma_{\mathfrak{p}} = \mathfrak{o}/\mathfrak{p}$ is a field. This field is called the *residue field* of the place $\mathfrak{p}$.

Lemma 1.3.5. *The valuation ring* $\mathfrak{o}$ *of* $\mathfrak{p}$ *contains an element* t *such that* $\mathfrak{p} = t\mathfrak{o}$ *and* $\bigcap_{n=1}^{\infty} t^n\mathfrak{o} = \{0\}$.

By Lemma 1.3.5, if $x \in L$, there exists at least one integer n such that $x \in t^n\mathfrak{o}$. In fact, if $x \in \mathfrak{o}$, we may take $n = 0$. If not, then x^{-1} is in $\mathfrak{o}$ and is $\neq 0$; therefore there exists an $m > 0$ such that

$$x^{-1} \in t^m\mathfrak{o}, x^{-1} \notin t^{m+1}\mathfrak{o}$$

which means that $t^{-m}x^{-1}$ is in $\mathfrak{o}$ but not in $t\mathfrak{o} = \mathfrak{p}$; i.e., $t^{-m}x^{-1}$ is a unit in $\mathfrak{o}$, and $x \in t^{-m}\mathfrak{o}$. If $x \neq 0$, there is by assumption a largest integer n such that $x \in t^n\mathfrak{o}$; denote by $v_{\mathfrak{p}}(x)$ this integer. If x, y are elements $\neq 0$ in $\mathfrak{o}$, then

$$v_{\mathfrak{p}}(x) + v_{\mathfrak{p}}(y) = v_{\mathfrak{p}}(xy) \tag{1.3}$$

and, if $x + y \neq 0$, then

$$v_{\mathfrak{p}}(x + y) \geq \min\{v_{\mathfrak{p}}(x), v_{\mathfrak{p}}(y)\}. \tag{1.4}$$

To complete the definition of the function $v_{\mathfrak{p}}$, we make the convention to write $v_{\mathfrak{p}}(0) = \infty$, where ∞ is a symbol with which we compute according to the following rules: $\infty > n$; $\infty + n = \infty$ for every integer n; $\infty \geq \infty$; $\infty + \infty = \infty$. Taking these conventions into account, the formulas (1.3) and (1.4) are valid in every case.

The function $v_{\mathfrak{p}}$ is called the *order function* at the place $\mathfrak{p}$; if $x \in L$, then $v_{\mathfrak{p}}(x)$ (i.e. $\mathrm{ord}_{\mathfrak{p}}(x)$ in [27]) is called the *order* of x at $\mathfrak{p}$. The elements of $\mathfrak{p}$ are the elements whose orders are > 0, and the units of $\mathfrak{o}$ are the elements of order 0. The elements t for which $t\mathfrak{o} = \mathfrak{p}$ are the elements of order 1; they are also called *local parameters*. The existence of places $\mathfrak{p}$ and order functions $v_{\mathfrak{p}}$ are ensured by [8, Theorem 1, page 6] and [8, Theorem 2, page 8], respectively.

Definition 1.3.6. Let $\mathfrak{p}$ be a place of L. If an element $x \in L$ belongs to $\mathfrak{p}$, then we say that $\mathfrak{p}$ is a *zero* of x; if $x^{-1} \in \mathfrak{p}$, then we say that $\mathfrak{p}$ is a *pole* of x. Furthermore, if there exists a value function $v_{\mathfrak{p}}$ at $\mathfrak{p}$, and $v_{\mathfrak{p}}(x) > 0$, then we say that $\mathfrak{p}$ is a zero of order $v_{\mathfrak{p}}(x)$ of x, while, if $v_{\mathfrak{p}}(x) < 0$, then we say that $\mathfrak{p}$ is a pole of order $-v_{\mathfrak{p}}(x)$ of x.

Definition 1.3.7. Let x be an element of L for which $\mathfrak{p}$ is not a pole. Then the residue class of x modulo $\mathfrak{p}$ (which is an element of the residue field $\Sigma_{\mathfrak{p}}$ of $\mathfrak{p}$) will be called the *value* taken by x at $\mathfrak{p}$. The value taken by x at $\mathfrak{p}$ is denoted by $x(\mathfrak{p})$; it is clear that, if neither x nor y has $\mathfrak{p}$ as a pole, then

$$(x + y)(\mathfrak{p}) = x(\mathfrak{p}) + y(\mathfrak{p}), \quad (xy)(\mathfrak{p}) = x(\mathfrak{p})y(\mathfrak{p}).$$

The elements which admit $\mathfrak{p}$ as a zero are those which take the value 0 at $\mathfrak{p}$.

Remark 1.3.8. It is often convenient to say an element of L which has $\mathfrak{p}$ as a pole takes the value ∞ at $\mathfrak{p}$; ∞ is here a symbol which has no intrinsic connection with the symbol ∞ which was used to complete the definition of the order function at a place.

Let L and E be two fields of algebraic functions of one variable such that L is a subfield of E. From [28, Chapter VII], E is an algebraic finite extension of L, and if $\mathfrak{o}$ is a valuation ring in L with maximal ideal $\mathfrak{p}$, then there exists a valuation ring $\mathfrak{D}$ in E, with prime $\mathfrak{B}$, such that $\mathfrak{o} = \mathfrak{D} \cap L \quad$ and $\quad \mathfrak{p} = \mathfrak{B} \cap L$. We say that the pair $(\mathfrak{D}, \mathfrak{B})$ *lies above* $(\mathfrak{o}, \mathfrak{p})$, or more briefly that $\mathfrak{B}$ lies above $\mathfrak{p}$. By [28, ChapterXII, Corollary 4.5], if $\mathfrak{o}$ is a valuation ring, then so is $\mathfrak{D}$. Therefore, $\Sigma_{\mathfrak{B}} = \mathfrak{D}/\mathfrak{B}$ and $\Sigma_{\mathfrak{p}} = \mathfrak{o}/\mathfrak{p}$ are residue fields, and $\Sigma_{\mathfrak{p}}$ can be seen as a subfield of $\Sigma_{\mathfrak{B}}$. If $\Sigma_{\mathfrak{B}}$ is algebraic of finite degree over $\Sigma_{\mathfrak{p}}$ then we say that $[\Sigma_{\mathfrak{B}} : \Sigma_{\mathfrak{p}}]$ is the *relative degree* of $\mathfrak{B}$. Let $v_{\mathfrak{B}}$ and $v_{\mathfrak{p}}$ are the value functions at the places $\mathfrak{B}$ and $\mathfrak{p}$ respectively. If u is a local parameter of $\mathfrak{B}$, then $t\mathfrak{D} = u^e\mathfrak{D}$, and e is called the *ramification index* of $\mathfrak{D}$ over $\mathfrak{o}$. It is clear that $v_{\mathfrak{B}}(x) = ev_{\mathfrak{p}}(x)$ for all $x \in L$. We say that $(\mathfrak{D}, \mathfrak{B})$ is *unramified* above $(\mathfrak{o}, \mathfrak{p})$, if the ramification index e is equal to 1.

Theorem 1.3.9. *[8, Theorem 1, page 52] Let L and E be fields of algebraic functions of one variable such that E contains L as a subfield. If $\mathfrak{p}$ is a place of L, then there exists at least one place of E which lies above $\mathfrak{p}$, and there exist only a finite number of such places, say $\mathfrak{B}_1, \cdots, \mathfrak{B}_h$. If E is of finite degree over L, the relative degrees $d_1, \cdots, d_h$ of $\mathfrak{B}_1, \cdots, \mathfrak{B}_h$ are finite, and we have*

$$[E : L] = \sum_{i=1}^{h} d_i e_i,$$

where e_i is the ramification index of $\mathfrak{B}_i$ with respect to L.

Let L be a field of algebraic functions of one variable over K.

Proposition 1.3.10. *([27, Proposition 1.2]) If $\mathfrak{o}_1$ and $\mathfrak{o}_2$ are two valuation rings with quotient field L, such that $\mathfrak{o}_1 \subset \mathfrak{o}_2$, then $\mathfrak{o}_1 = \mathfrak{o}_2$.*

By a *point* of L over K, we shall mean a valuation ring $\mathfrak{o}$ of L. By Proposition 1.3.10 the valuation rings $\mathfrak{o}_i$ $(i = 1, \ldots, n)$ of L are distinct and have no inclusion relations. The set of all points of L will be called a *curve*, denoted by C, whose function field is L. We use the letters P, Q for points of the curve C.

Definition 1.3.11. A *divisor* (on the curve C, or of L over K) is an element of the free abelian group generated by the points. Thus a divisor is a formal sum

$$\mathfrak{a} = \sum_{P \in K} n_P P$$

where P are points, and n_P are integers, all but a finite number of which are 0. We call $\sum n_P$ the *degree* of $\mathfrak{a}$, and n_P the *order* of $\mathfrak{a}$ at P.

If $x \in L\backslash\{0\}$, there is only a finite number of points P such that

$$\mathrm{ord}_P(x) := v_{\mathfrak{p}}(x) \neq 0.$$

Indeed by the formulas (1.3) and (1.4), if x is constant, then $\mathrm{ord}_P(x) = 0$ for all P. If x is not constant, then there is one point of $K(x)$ at which x has a zero, and one point at which x has a pole. Each of these points extends to only a finite number of points of L, which is a finite extension of $K(x)$. Hence we can associate a divisor with x, namely

$$(x) = \sum_{\mathrm{ord}_P(x) \neq 0} n_P P,$$

where $n_P = \mathrm{ord}_P(x)$. If $\mathfrak{a} = \sum n_P P$ and $\mathfrak{b} = \sum m_P P$ are divisors, we write $\mathfrak{a} \geq \mathfrak{b}$ if and only if $n_P \geq m_P$ for all P. We call $\mathfrak{a}$ *positive* if $\mathfrak{a} \geq 0$.

If $\mathfrak{a}$ is a divisor, we denote by $\mathfrak{L}(\mathfrak{a})$ the set of all elements $x \in L$ such that $(x) \geq -\mathfrak{a}$. If $\mathfrak{a}$ is a positive divisor, then $\mathfrak{L}(\mathfrak{a})$ consists of all the functions in L which have poles only in $\mathfrak{a}$, with multiplicities at most those of $\mathfrak{a}$. Since

$$\mathrm{ord}_P(x+y) \geq \min\{\mathrm{ord}_P(x), \mathrm{ord}_P(y)\}$$

and

$$\mathrm{ord}_P(\lambda x) = \mathrm{ord}_P(x)$$

for all P and $\lambda \in K$, it is clear that $\mathfrak{L}(\mathfrak{a})$ is a vector space over K, see Lemma 1.3.12.

Lemma 1.3.12. *$\mathfrak{L}(\mathfrak{a})$ is a vector space over the field K for any divisor $\mathfrak{a}$. Let $l(\mathfrak{a})$ be the dimension of $\mathfrak{L}(\mathfrak{a})$. If $\mathfrak{a} \geq \mathfrak{b}$ then $\mathfrak{L}(\mathfrak{a}) \subseteq \mathfrak{L}(\mathfrak{a})$ and $l(\mathfrak{b}) \leq l(\mathfrak{a})$.*

Let P be a point of C and $\mathfrak{o}$ be its local ring of L. Let $\mathfrak{p}$ be its maximal ideal. Since K is algebraically closed, $\Sigma_{\mathfrak{p}} = \mathfrak{o}/\mathfrak{p}$ is canonically isomorphic to K. Let t be a generator (i.e. local parameter) of $\mathfrak{p}$ and x be an element of $\mathfrak{o}$. Then for some constant $a_0 \in K$, we can write

$$x \equiv a_0 \bmod \mathfrak{p}.$$

The function $(x - a_0)$ is in $\mathfrak{p}$, and it has a zero in $\mathfrak{o}$. We can therefore write

$$x - a_0 = t y_0, \text{ where } y_0 \in \mathfrak{o}.$$

Again by a similar argument we get $y_0 = a_1 + t y_1$ with $y_1 \in \mathfrak{o}$, and

$$x = a_0 + a_1 t + y_1 t^2.$$

Continuing this procedure, we obtain an expansion of x into a power series,

$$x = a_0 + a_1 t + a_2 t^2 + \cdots.$$

It is trivial that if each coefficient a_i is equal to 0, then $x = 0$. The quotient field L of $\mathfrak{o}$ can be embedded in the power series field $K((t))$ (see [8, 59]) as follows. If x is in L, then for some power t^s, the function $t^s x$ lies in $\mathfrak{o}$, and hence x can be written

$$x = \frac{a_s}{t^s} + \frac{a_{s-1}}{t^{s-1}} + \cdots \frac{a_{-1}}{t} + a_0 + a_1 t + \cdots.$$

If u is another generator of $\mathfrak{p}$, then clearly $K((t)) = K((u))$, and our power series field depends only on P. We denote it by L_P. An element $\xi_P \in L_P$ can be written

$$\xi_P = \sum_{v=m}^{\infty} a_v t^v$$

with $a_m \neq 0$. If $m < 0$, we say that ξ_P has a *pole* of order $-m$. If $m > 0$ we say that ξ_P has a *zero* of order m, and we let $m = \mathrm{ord}_P(\xi_P)$.

Lemma 1.3.13. *For any divisor $\mathfrak{a}$ and any point P, we have $l(\mathfrak{a})$ is finite and*

$$l(\mathfrak{a} + P) \leq l(\mathfrak{a}) + 1.$$

Let A^* be the cartesian product of all L_P, taken over all points P. An element of A^* can be viewed as an infinite vector $\xi = (\ldots, \xi_P, \ldots)$ where $\xi_P \in L_P$. The selection of such an element in A^* means that a random power series has been selected at each point P. Under component wise addition and multiplication, A^* is a ring. Let subring A of A^* consisting of all vectors such that ξ_P has no pole at P for all but a finite number of P. This ring A will be called the ring of *adeles*. Note that the function field L is embedded in A under the mapping

$$x \to (\ldots, x, x, x, \ldots),$$

i.e., at the P-component we take x viewed as a power series in L_P.

Let $\mathfrak{a}$ be a divisor. We shall denote by $\Lambda(\mathfrak{a})$ the subset of A consisting of all adeles ξ such that

$$\mathrm{ord}_P(\xi_P) \geq -\mathrm{ord}_P(\mathfrak{a}).$$

Then $\Lambda(\mathfrak{a})$ is immediately seen to be a K-subspace of A since

$$\mathrm{ord}_P(\xi_P + \eta_P) \geq \min(\mathrm{ord}_P(\xi_P), \mathrm{ord}_P(\eta_P)) \geq -\mathrm{ord}_P(\mathfrak{a}),$$

and

$$\mathrm{ord}_P(\lambda\xi_P) = \mathrm{ord}_P(\xi_P) \geq -\mathrm{ord}_P(\mathfrak{a}), \forall \lambda \in k.$$

The set of functions $x \in L$ such that $(x) \geq -\mathfrak{a}$ is the vector space $\mathfrak{L}(\mathfrak{a})$, and is immediately seen to be equal to $\Lambda(\mathfrak{a}) \cap L$. If B and C are two K-subspaces of A, and $B \supset C$, then we denote by $(B : C)$ the dimension of the factor space B mod C over K.

Proposition 1.3.14. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two divisors. Then $\Lambda(\mathfrak{a}) \supset \Lambda(\mathfrak{b})$ if and only if $\mathfrak{a} \geq \mathfrak{b}$. If this is the case, then*

1. $(\Lambda(\mathfrak{a}) : \Lambda(\mathfrak{b})) = deg(\mathfrak{a}) - deg(\mathfrak{b})$, *and*

2. $(\Lambda(\mathfrak{a}) : \Lambda(\mathfrak{b})) = ((\Lambda(\mathfrak{a}) + L) : (\Lambda(\mathfrak{b}) + L)) + ((\Lambda(\mathfrak{a}) \cap L) : (\Lambda(\mathfrak{b}) \cap L))$.

From Proposition 1.3.14, we get the fundamental formula:

$$\deg(\mathfrak{a}) - \deg(\mathfrak{b}) = (\Lambda(\mathfrak{a}) + L : \Lambda(\mathfrak{b}) + L) + l(\mathfrak{a}) - l(\mathfrak{b}) \tag{1.5}$$

for two divisors $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a} \geq \mathfrak{b}$.

Definition 1.3.15. Divisors $\mathfrak{a}$ and $\mathfrak{b}$ are said to be *linearly equivalent* if $\mathfrak{a} - \mathfrak{b}$ is the divisor of a function. A function depending only on linear equivalence will be called a *class function*.

Lemma 1.3.16. *Both* $\deg(\mathfrak{a})$ *and* $l(\mathfrak{a})$ *are class functions.*

By Lemma 1.3.16, the new function defining by

$$r(\mathfrak{a}) = \deg(\mathfrak{a}) - l(\mathfrak{a})$$

is a class function. We need to prove $r(\mathfrak{a})$ is bound for any divisor $\mathfrak{a}$.

Proposition 1.3.17. *For any divisor* $\mathfrak{a}$*, the integer* $r(\mathfrak{a})$ *is bound. Moreover, there exists a divisor* $\mathfrak{a}$ *such that* $A = \Lambda(\mathfrak{a}) + L$.

This result allows us to split the index in (1.5). We denote the dimension of $(A : \Lambda(\mathfrak{a}) + L)$ by $\delta(\mathfrak{a})$, which is finite by Proposition 1.3.17, and (1.5) becomes

$$\deg(\mathfrak{a}) - \deg(\mathfrak{b}) = \delta(\mathfrak{b}) - \delta(\mathfrak{a}) + l(\mathfrak{a}) - l(\mathfrak{b}) \tag{1.6}$$

or in other words

$$l(\mathfrak{a}) - \deg(\mathfrak{a}) - \delta(\mathfrak{a}) = l(\mathfrak{b}) - \deg(\mathfrak{b}) - \delta(\mathfrak{b}). \tag{1.7}$$

This holds for $\mathfrak{a} \geq \mathfrak{b}$. However, since two divisors have a sup, (1.7) holds for any two divisors $\mathfrak{a}$ and $\mathfrak{b}$. By (1.7), we have the following definition.

Definition 1.3.18. The *genus* of L is defined to be that integer g such that

$$l(\mathfrak{a}) - \deg(\mathfrak{a}) - \delta(\mathfrak{a}) = 1 - g.$$

By the formula (1.7), g is an invariant of L. Putting $\mathfrak{a} = 0$, we obtain $g = \delta(0)$. Hence $g = (A : \Lambda(0) + L)$, which is an integer ≥ 0. Finally, we have the following result.

Definition 1.3.19. There exists an integer $g \geq 0$ depending only on L such that for any divisor $\mathfrak{a}$ we have

$$l(\mathfrak{a}) = \deg(\mathfrak{a}) + 1 - g + \delta(\mathfrak{a}), \delta(\mathfrak{a}) \geq 0.$$

The following formula compares the genus of a finite extension, in terms of the ramification indices. The details of its proof can be found in [27, page 27].

Theorem 1.3.20. *(The Genus Formula of Hurwitz) Let K be an algebraically closed field, and let L be a function field with K as constant field. Let E be a finite separable extension of L of degree n. Let g_E and g_L be the genera of E and L respectively. For each point P of L, and each point Q of E above P, assume that the ramification index e_Q is prime to the characteristic of K. Then*

$$2g_E - 2 = n(2g_L - 2) + \sum_Q (e_Q - 1). \tag{1.8}$$

We conclude this section by giving two well-known models of fields of algebraic functions which can be easily found in literature, for instance see [8, 26]. First, we are going to investigate the places (and also the valuation rings) of $K(x)$. From that, we can see something occurring in a field of algebraic functions L since it is algebraic of finite degree over $K(x)$.

Example 1.3.21. Let K be an algebraically closed field and x transcendental over K. Let $a \in K$ and let $\mathfrak{o}_a$ be the set of rational functions

$$\mathfrak{o}_a = \left\{ \frac{f(x)}{g(x)} \mid f(x), g(x) \in K[x], g(a) \neq 0 \right\}.$$

Then $\mathfrak{o}_a$ is a valuation ring, whose maximal ideal $\mathfrak{p}_a$ consists of all such quotients such that $f(a) = 0$. Let $\mathfrak{o}$ be a discrete valuation ring in $K(x)$ containing K, and $\mathfrak{p}$ be its maximal ideal. Then $\mathfrak{p} \cap K[x] \neq 0$, and $\mathfrak{p} \cap K[x]$ is therefore generated by an irreducible polynomial $p(x)$, which must be of degree 1 since we assumed K algebraically closed. Thus $p(x) = x - a$ for some $a \in K$. Then it is clear that the canonical map

$$\mathfrak{o} \to \mathfrak{o}/\mathfrak{p}$$

induces the map

$$f(x) \to f(a)$$

on polynomials, and it follows that $\mathfrak{o}$ consists of all quotients $\dfrac{f(x)}{g(x)}$ such that $g(a) \neq 0$. Hence, such $\mathfrak{o}$ is of the form $\mathfrak{o}_a$ and $\mathfrak{p} = \mathfrak{p}_a = (x - a)\mathfrak{o}_a$. Clearly that $\mathfrak{p}_a \neq \mathfrak{p}_b$ if $a \neq b$. Let $x^* = x^{-1}$, then $K(x) = K(x^*)$. We denote $\mathfrak{p}_{1/x}$ for the place of x^* and $\mathfrak{o}_{1/x}$ is the ring of this place. Then $\mathfrak{p}_{1/x}$ is distinct from all the place $\mathfrak{p}_a$ above since $x \in \mathfrak{o}_a$ but $x \notin \mathfrak{o}_{1/x}$. Since $\mathfrak{o}_{1/x}$ does not correspond with an element $a \in K$, we can say it is valuation ring of the infinitive point. Finally, the set of $\mathfrak{o}_a$ and $\mathfrak{o}_{1/x}$ exhaust all valuation rings of $K(x)$.

Theorem 1.3.22. *Let K be an algebraically closed field and L be a field of algebraic functions of one variable over K, then L has genus zero if an only if $L = K(x)$.*

Theorem 1.3.23. *Let $L = K(x, y)$ where y satisfies the equation*

$$y^2 = f(x), \tag{1.9}$$

and $f(x)$ is a polynomial of degree n, with distinct roots. Then the genus of L is

$$\left[\frac{n-1}{2}\right].$$

Such L is called hyperelliptic function field.

Proof. Let

$$f(x) = \prod_{i=1}^{n}(x - a_i),$$

where the elements a_i are distinct. By classified as in Example 1.3.21, all points P of $K(x)$ correspond with the set of valuation rings $\mathfrak{o}_a$ (where $a \in K$), and the infinity point ∞ corresponds with $\mathfrak{o}_{1/x}$. Since $(x - a)$ is a unit when we consider at point P_i which corresponds with $(x - a_i)$, then L is unramified over $K(x)$ at all points except the points P_i (corresponding to $x = a_i$), and also possibly at those points lying above $x = \infty$. At P_i the ramification index is 2. Let $t = 1/x$ so that t has order 1 at ∞ in $K(x)$. We write

$$f(x) = \prod_{i=1}^{n} t^{-n}(1 - ta_i).$$

Since $\prod_{i=1}^{n}(1 - ta_i)$ is a unit when we consider at t, then $f(x)$ has a pole of order n at $t \in K(t)$. That means $f(x)$ has a zero of order n at the infinity point ∞ of $k(x)$. By Theorem 1.3.22, $K(x)$ has genus zero. If n is odd, by equation (1.9), $K(x, y)$ is ramified of order 2 at infinity and the Hurwitz genus formula yields

$$2g_L - 2 = 2(2 \cdot 0 - 2) + \sum_{i=1}^{n}(2 - 1) + (2 - 1) = n - 3,$$

which follows

$$g_L = \frac{n-1}{2}.$$

If n is even, then the ramification index at infinity is 1 and the Hurwitz formula yields

$$g_L = \frac{n-2}{2}.$$

The proof is complete. □

1.4 Rational functions on algebraic curves

In this section, we introduce the models of fields of algebraic functions related to projective algebraic curves.

Definition 1.4.1. The *field of rational functions* on $\mathbb{P}^2(K)$ is the set of all quotients

$$\frac{\phi}{\psi} \in K(X_0, X_1, X_2),$$

where $\phi, \psi \in K[X_0, X_1, X_2]$ are relatively prime, homogeneous of the same degree, $\psi \neq 0$. Such a fraction $\frac{\phi}{\psi}$ gives a function r that vanishes on $\mathcal{V}(\phi)$ as follows

$$r : \mathbb{P}^2(K) \backslash \mathcal{V}(\psi) \to K \ (P \to \frac{\phi(P)}{\psi(P)}).$$

We call r a *rational function* on the projective plane whose domain of definition $\mathrm{Def}(r)$ is $\mathbb{P}^2(K)\backslash\mathcal{V}(\psi)$. We call ψ the *pole divisor* and ϕ the *zero divisor* of the rational function r. We shall write $\mathcal{R}(\mathbb{P}^2)$ for the field of rational functions on $\mathbb{P}^2(K)$. The field K is embedded into $\mathcal{R}(\mathbb{P}^2)$ as the field of constant functions.

Definition 1.4.2. For $P \in \mathbb{P}^2(K)$, we denote $\mathcal{O}_P$ by

$$\mathcal{O}_P = \left\{ \frac{\phi}{\psi} \in \mathcal{R}(\mathbb{P}^2) \mid \psi(P) \neq 0 \right\}.$$

We call $\mathcal{O}_P$ the *local ring* of P on $\mathbb{P}^2(K)$. The maximal ideal of $\mathcal{O}_P$ is

$$\mathbf{m}_P = \left\{ \frac{\phi}{\psi} \in \mathcal{O}_P \mid \phi(P) = 0 \right\}.$$

Lemma 1.4.3. *([26, Lemma 4.1]) Dehomogenization gives a K-isomorphism*

$$\rho : \mathcal{R}(\mathbb{P}^2) \xrightarrow{\simeq} K(X,Y) \left(\frac{\phi}{\psi} \to \frac{\phi(1,X,Y)}{\psi(1,X,Y)} \right).$$

Let $P = (a, b)$ be a point at finite distance and $\mathcal{M}_P = (X - a, Y - b)$ be its maximal ideal in $K[X, Y]$, then ρ induces a K-isomorphism

$$\mathcal{O}_P \xrightarrow{\simeq} K[X,Y]_{\mathcal{M}_P}$$

onto the localization of $K[X, Y]$ with respect to $\mathcal{M}_P$.

The elements of $K(X, Y)$ can be treated of as functions on $\mathbb{A}^2(K)$, and ρ assigns to each rational function on $\mathbb{P}^2(K)$ its restriction to $\mathbb{A}^2(K)$. We call $\mathcal{R}(\mathbb{A}^2) = K(X, Y)$

the *field of rational functions* on $\mathbb{A}^2(K)$. Let $\hat{F}$ be defining homogeneous polynomial of a projective curve $\Gamma \subset \mathbb{P}^2(K)$ of positive degree. Then

$$\mathcal{O}_{\hat{F}} := \left\{ \frac{\phi}{\psi} \in \mathcal{R}(\mathbb{P}^2) \mid \hat{F} \text{ and } \psi \text{ are relatively prime} \right\}$$

is a subring of $\mathcal{R}(\mathbb{P}^2)$ and

$$\mathcal{I}_{\hat{F}} := \left\{ \frac{\phi}{\psi} \in \mathcal{R}(\mathbb{P}^2) \mid \phi \in \left\langle \hat{F} \right\rangle \right\}$$

is an ideal of $\mathcal{O}_{\hat{F}}$. The ring $\mathcal{O}_{\hat{F}}$ consists of precisely rational functions that are defined on $\mathcal{V}(\hat{F})$ up to a finite set of exceptions and $\mathcal{I}_{\hat{F}}$ consists of the functions that vanishing on $\mathcal{V}(\hat{F})$.

Definition 1.4.4. The residue class ring

$$R(\hat{F}) := \mathcal{O}_{\hat{F}} / \mathcal{I}_{\hat{F}}$$

is called the *ring of rational functions* on $\hat{F}$ (i.e. on Γ).

Definition 1.4.5. We call two following residue class rings

$$K[F] := K[X, Y]/(F) \text{ and } K[\hat{F}] := K[X_0, X_1, X_2]/(\hat{F})$$

the *affine coordinate ring* of F and *projective coordinate ring* of $\hat{F}$, respectively.

Theorem 1.4.6 follows from [26, Theorem 4.4] and [26, Corollary 4.6].

Theorem 1.4.6. *Suppose that $\hat{F}$ is an irreducible homogeneous polynomial of positive degree in $K[X_0, X_1, X_2]$ and F is the affine curve associated with $\hat{F}$; suppose that $K[F]$ is its coordinate ring, and $Q(K[F])$ is the full ring of quotients of $K[F]$. Then the followings hold.*

1. *$R(\hat{F})$ is a field.*
2. *There is a $K-$isomorphism $\mathcal{R}(\hat{F}) \xrightarrow{\simeq} Q(K[F])$. This follows the ring $Q(K[F])$ is also a field.*
3. *If x and y denote the residue classes of X and Y in $K[F]$, then*

$$\mathcal{R}(\hat{F}) \xrightarrow{\simeq} K(x, y).$$

In this case x (without loss of generality) is transcendental over K, and $\mathcal{R}(\hat{F})$ is a separable algebraic extension of $K(x)$.

Remark 1.4.7. From Theorem 1.4.6, every field of algebraic functions of one variable L (Definition 1.3.1) is $K-$isomorphic to the field $\mathcal{R}(\hat{F})$ of rational functions of a suitably chosen irreducible algebraic curve $\Gamma = \mathcal{V}(\hat{F})$. Such Γ is called a *projective curve model* of L. From [26, Chapter 14], if Γ is smooth, then the genus of Γ is equal to the genus of $\mathcal{R}(\hat{F})$. In addition, the genus of $\mathcal{R}(\hat{F})$ is of zero if and only if so is the genus of Γ.

Example 1.4.8. The Fermat curve Γ defined by

$$\hat{F} = X_1^n + X_2^n - X_0^n, n \geq 3$$

is a smooth curve since it has no singularity. The genus of Γ is

$$g(\Gamma) = \frac{(n-1)(n-2)}{2}.$$

Let F be the dehomogenization of $\hat{F}$ as follows

$$F(X,Y) = \hat{F}(1,X,Y) = X^n + Y^n - 1,$$

then it is the defining polynomial of the Fermat curve $\mathcal{C}$ in the affine plane. A field of algebraic functions L over K with an affine curve model $\mathcal{C}$ is isomorphic to $\mathcal{R}(\hat{F})$. In this case, the genus of $\mathcal{R}(\hat{F})$ is equal to $g(\Gamma)$.

Example 1.4.9. A function field L is called *hyperelliptic* if it has as a model an affine curve with equation

$$F := Y^2 - P(X),$$

where $P(X) \in K[X]$ is a polynomial of degree $n \geq 3$ with distinct roots. We call the projective closure Γ defined by $\hat{F}$ a *hyperelliptic curve*. By Theorem 1.3.23

$$g_L = \begin{cases} \dfrac{n}{2} - 1 & \text{if } n \text{ is even;} \\ \dfrac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

If $n = 3$, then L is an elliptic function field which has genus one. In this case, Γ is a non-singular curve. We note that, if $n \geq 4$, then Γ is not a smooth curve (non-singular). In fact, if it is the case then by Theorem 1.4.6, we have the following contradiction

$$\left[\frac{n-1}{2}\right] = \frac{(n-1)(n-2)}{2}, n \geq 4.$$

Hence, Γ is not a smooth curve. However, from the computation in [17, Chapter 10], $g(\Gamma)$ is equal to g_L.

1.5 Preparation

This section prepares some main tools which are regularly used in the dissertation.

Definition 1.5.1. Let $F(y, y') = 0$ be a first-order AODE over K. The algebraic curve $F(y, w) = 0$ where $F(y, w) \in K[y, w]$ is said to be the *corresponding algebraic curve* of the AODE $F(y, y') = 0$.

By Definition 1.5.1, we may consider a first-order AODE via an algebraic curve. From that, some important concepts related to algebraic curves such as associated fields of algebraic functions and rational parametrizations (of algebraic curves of genus zero) are introduced in Section 1.5.1 and Section 1.5.2, respectively.

1.5.1 Associated fields of algebraic functions

Due to Theorem 1.4.6 and Remark 1.4.7, Definition 1.5.2 gives a connection between an affine algebraic curve and a field of algebraic functions in Section 1.3.

Definition 1.5.2. Assume that L is a field of algebraic functions over K, then there are $\eta, \xi \in L$ such that $L = K(\eta, \xi)$, where η is transcendental over K and ξ is algebraic over $K(\eta)$. The function field $L = K(\eta, \xi)$ is called an *associated field of algebraic functions* of the affine algebraic curve $\mathcal{C}$ defined by irreducible polynomial F if $F(\eta, \xi) = 0$. Such $\mathcal{C}$ is called the *affine algebraic curve model* of the function field L.

Lemma 1.5.3 has an intrinsic role for the problems of solving first-order AODEs of genus zero considered in Chapter 3 and Section 4.1.

Lemma 1.5.3. *[37, Lemma 2.7] If η is a solution of the AODE $F(y, y') = 0$ which is transcendental over K, then $K(\eta, \eta')$ is an associated field of algebraic functions of the corresponding algebraic curve $\mathcal{C}$ defined by $F(y, w)$. In addition, if $\mathcal{C}$ is of genus zero then its associated field of algebraic functions $K(\eta, \eta')$ is of the form $K(t)$.*

Proof. From Definition 1.5.2, if η is a transcendental solution (not necessary liouvillian) over K of $F(y, y') = 0$ then $K(\eta, \eta')$ is an associated field of algebraic functions of $\mathcal{C}$. Moreover, $\mathcal{C}$ has genus zero means that its projective closure Γ defined by $\hat{F}$ is of genus zero, see Definition 1.2.6. From Theorem 1.4.6 and Remark 1.4.7, then the field $K(\eta, \eta')$ is of genus zero. From Theorem 1.3.22, $K(\eta, \eta')$ is of the form $K(t)$. □

Remark 1.5.4. The associated fields of algebraic functions in Lemma 1.5.3 have been studied in [23, 30, 31]. In a special case, if $K = \overline{\mathbb{Q}}$, then they coincide with the ones considered in [1, Definition 3.2].

1.5.2 Rational parametrizations

Definition 1.5.5. A *rational parametrization* of an algebraic curve $\mathcal{C}$ defined by an irreducible polynomial $F(y,w)$ is a pair of rational functions $\mathcal{P}(t) = (r(t), s(t)) \in K(t)^2$ such that the two following items hold.

1. For almost all t_0 the point $\mathcal{P}(t_0) = (r(t_0), s(t_0)) \in \mathcal{C}$.

2. For almost all point $(x_0, y_0) \in \mathcal{C}$ there exists $t_0 \in K$ such that $\mathcal{P}(t_0) = (x_0, y_0)$.

An algebraic curve $\mathcal{C}$ is said to be *rational* or a *rational curve* if it admits a rational parametrization $\mathcal{P}(t)$. Moreover, if t_0 is unique then such $\mathcal{P}(t)$ is said to be *proper* or a *proper parametrization* of $\mathcal{C}$.

Lemma 1.5.6. *([50, Lemma 4.13]) Every rational curve can be properly parametrized.*

Theorem 1.5.7. *[50, Theorem 4.14] Let $\mathcal{P}(t) = (r(t), s(t))$ be a rational parametrization of an algebraic curve $\mathcal{C}$ defined by $F(y,w) = 0$. Then, the following statements are equivalent:*

1. *$\mathcal{P}(t) = (r(t), s(t))$ is proper.*

2. *$K(\mathcal{P}(t)) = K(t)$.*

The following lemma gives a relation between two different parametrizations of an algebraic curve of genus zero.

Lemma 1.5.8. *([50, Lemma 4.17]) Let $\mathcal{P}(t)$ be a proper parametrization of a rational curve $\mathcal{C}$, and $\tilde{\mathcal{P}}(t)$ is any rational parametrization of $\mathcal{C}$. Then there is a non-constant rational function $\varphi(t) \in K(t)$ such that $\tilde{\mathcal{P}}(t) = \mathcal{P}(\varphi(t))$. Moreover, $\tilde{\mathcal{P}}(t)$ is a proper parametrization if and only if there is a linear function $\varphi(t) = \dfrac{at+b}{ct+d} \in K(t)$ such that $\tilde{\mathcal{P}}(t) = \mathcal{P}(\varphi(t))$.*

Finally, Theorem 1.5.9 shows that only algebraic curves of genus zero are rational.

Theorem 1.5.9. *[50, Theorem 4.63] An algebraic curve is rational if and only if its genus is equal to zero.*

Conclusion

In this chapter, we introduce the terminologies and the main tools which are used in the dissertation. In addition, we try to outline their chemistry as far as we can. We hope this may provide the readers the essential information when reading the text.

Chapter 2

Rational liouvillian solutions of first-order autonomous AODEs

The content of this chapter is mainly based on the author's works in [35]. In this chapter, we study rational liouvillian general solutions of first-order autonomous AODEs

$$F(y, y') = 0, \tag{2.1}$$

where $F(y, w) \in \mathbb{C}[y, w]$ is an irreducible polynomial. We define rational liouvillian solutions by using the criterion of liouvillian solution in [56] and then tailor the method of Feng and Gao in [14] to the case of finding rational liouvillian solutions of first-order AODEs (2.1). Such a rational liouvillian solution induces a rational proper parametrization. This follows the necessary and sufficient conditions of such differential equation for having a rational liouvillian solution over $\mathbb{C}$. The results turn out to be an algorithm for determining a rational liouvillian solution the AODE (2.1).

This chapter is organized as follows. In Section 2.1, we present the main idea in [14] for determining rational general solutions of the AODE (2.1), and we also introduce its extension cases for finding algebraic and radical general solutions, see [1] and [18], respectively. In Section 2.2, we define rational liouvillian solutions and separate them from others. Section 2.3 gives necessary and sufficient conditions for the AODE (2.1) having a non-constant rational liouvillian solution. An algorithm and some examples are presented in Section 2.4.

2.1 Solving first-order AODEs by parametrizations

The general idea of the method which is called the *algebro-geometric method* is to associate first-order AODEs to geometry objects, i.e. algebraic curves or algebraic surfaces, then use their geometric properties which also satisfy the derivative constrain to deduce the solutions of the original differential equations. There are notable works (not exhausted) in the method such as [14, 33, 57]. A summarization of more aspects of the algebro-geometric method can be found in [12]. In this section, we present the method of [14] and the ideas of [1] for solving first-order autonomous AODEs.

First, we recall the method in [14] for finding rational general solution of a first-order autonomous AODE (2.1). The idea is to associate such AODE to an algebraic curve defined by an irreducible polynomial $F(y, w) \in \mathbb{Q}[y, w]$, see Definition 1.5.1, and then use the rational parametrizations of the algebraic curve which satisfies the derivative constrain to find a rational general solution.

Definition 2.1.1. ([14, Definition 1]) A rational solution of the AODE (2.1) is defined as a solution (see Chapter 1) of such AODE of the form

$$y(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}, \tag{2.2}$$

where $m, n \in \mathbb{N}$, a_i, b_j are in a universal constant extension of $\mathbb{Q}$. It is called a *non-trivial* solution if $\deg_x y(x) > 0$.

Lemma 2.1.2. *([14, Lemma 4]) Assume that a first-order AODE (2.1) has a non-trivial rational solution*

$$y(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}, \tag{2.3}$$

where $m, n \in \mathbb{N}$, $a_i, b_j \in \mathbb{Q}, a_n \neq 0$. Then $y(x+c)$ is a general solution *of the differential equation (2.1) for an arbitrary constant c.*

If $y(x)$ is a non-trivial rational solution of the AODE (2.1), then the pair

$$(y(x), y'(x))$$

can be regarded as a rational parametrization of the algebraic curve $F(y, w) = 0$. The following theorem shows that such parametrization is proper, see Theorem 1.5.7.

Theorem 2.1.3. *([14, Theorem 2]) Assume that the differential equation (2.1) has a non-trivial rational solution $y(x)$. Then $\bar{\mathbb{Q}}(y(x), y'(x)) = \bar{\mathbb{Q}}(x)$.*

There is a criterion for the AODE (2.1) having a rational general solution.

Theorem 2.1.4. *([14, Theorem 5]) Let $y = r(x), w = s(x)$ be a proper parametrization of $F(y, w) = 0$, where $r(x), s(x) \in \bar{\mathbb{Q}}(x)$. Then $F = 0$ has a rational general solution if and only if*

$$ar'(x) = s(x) \quad \text{or} \quad a(x-b)^2 r'(x) = s(x) \tag{2.4}$$

where $a, b \in \bar{\mathbb{Q}}$ and $a \neq 0$. If one of the above relations is true, then replacing x by $a(x+c)$ (or $b - \frac{1}{a(x+c)}$) in $y = r(x)$, we obtain a rational general solution of $F = 0$, where c is an arbitrary constant.

Proof. Let $y = q(x)$ be a non-trivial rational solution of $F = 0$. From Theorem 2.1.3, $(q(x), q'(x))$ is also a proper parametrization of $F(y, w) = 0$. By Lemma 1.5.8, there is

$$\varphi(x) = \frac{c_1 x + c_2}{c_3 x + c_4}, c_1 c_4 - c_2 c_3 \neq 0$$

such that

$$\begin{aligned} q(x) &= r(\varphi(x)) \\ q'(x) &= s(\varphi(x)) = (r(\varphi(x)))' = r'(\varphi(x))\varphi'(x). \end{aligned} \tag{2.5}$$

If $c_3 = 0$ then $\varphi(x) = \frac{c_1 x + c_2}{c_4}$. This implies $\varphi'(x) = \frac{c_1}{c_4}$. From (2.5), we obtain $s(\varphi(x)) = ar'(\varphi(x))$, where $a = \frac{c_1}{c_4}$. If $c_3 \neq 0$, then

$$\varphi'(x) = \frac{c_1 c_4 - c_2 c_3}{(c_3 x + c_4)^2}.$$

By replacing

$$x = \frac{1}{(c_1 c_4 - c_2 c_3)} \frac{c_4 \varphi(x) - c_2}{-c_3 \varphi(x) + c_4}$$

we obtain

$$\varphi'(x) = \frac{c_3^2 (\varphi(x) - c_1/c_3)^2}{c_1 c_4 - c_2 c_3}.$$

Also from (2.5), we have

$$s(\varphi(x)) = a(\varphi(x) - b)^2 r'(\varphi(x)),$$

where $a = \frac{c_3^2}{c_1 c_4 - c_2 c_3}$ and $b = c_1/c_3$. In both cases, we obtain a rational solution of $F = 0$, which is $q(x) = r(\varphi(x))$. From Lemma 2.1.2, the rational general solution of $F = 0$ is $q(x+c)$. If the condition (2.4) holds, the reverse is easy to follow. In fact, let $q(x) = r(\varphi(x))$, then from (2.5) we have

$$q'(x) = (r(\varphi(x)))' = s(\varphi(x)),$$

which follows that $F(q(x), q'(x)) = 0$. Hence, $q(x)$ is a rational solution of $F = 0$. □

These results lead to an algorithm, i.e. [14, Algorithm 1], for finding a rational general solution of a first-order autonomous AODE. In addition, we give an example to illustrate how the algorithm works.

Algorithm `RatSol`

Input: An algebraic curve $F(y, w) = 0$ over $\mathbb{Q}$.

Output: A rational general solution of $F(y, y') = 0$ if any.

---

1. If the algebraic curve $F(y, w) = 0$ is not rational, then **return** "$F(y, y') = 0$ has no rational general solutions".
2. Else, compute a proper parametrization $(r(x), s(x))$ of $F(y, w) = 0$.
3. Let $A = \dfrac{s(x)}{r'(x)}$.

   (a) If $A = a \in \mathbb{Q}$, substituting x by $a(x + c)$ in $r(x)$, then $F(y, y') = 0$ has a rational general solution $y = r(a(x + c))$.

   (b) If $A = a(x - b)^2$ for $a, b \in \mathbb{Q}$, substituting x by $\dfrac{ab(x + c) - 1}{a(x + c)}$ in $r(x)$, then $F(y, y') = 0$ has a rational general solution $y = r(\dfrac{ab(x + c) - 1}{a(x + c)})$.

   (c) Otherwise, by Theorem 2.1.4, the algorithm terminates, then **return** "$F(y, y') = 0$ has no rational general solutions".

Remark 2.1.5. Algorithm `RatSol` depends on the rational parametrizations of a plane algebraic curve, which is computationally difficult in the early years of 2000s. Noting that the problem of finding rational parametrizations of an algebraic curve has been solved completely, see [50], and now with the helping of MAPLE, the algorithm can be easily processed. By Theorem 2.1.3 and Lemma 1.5.8, the rational general solution of the form $r(x)$ in Theorem 2.1.4 covers all of the rational general solutions of the AODE (2.1), hence, Algorithm `RatSol` is complete.

Example 2.1.6. (see [14, Example 1]) Consider first-order autonomous AODE

$$F(y, y') = y'^3 + 4y'^2 + (27y^2 + 4)y' + 27y^4 + 4y^2 = 0.$$

By MAPLE, the corresponding curve $F(y, w) = 0$ has a proper parametrization

$$(r(x), s(x)) = \left(\frac{-4x^2 - 64}{x^3}, \frac{-16x^2 - 768}{x^4}\right).$$

We obtain $\dfrac{s(x)}{r'(x)} = -4$, then $a = -4$ and $b = 0$. From Theorem 2.1.4,

$$y = r(-4(x+c)) = \frac{(x+c)^2+1}{(x+c)^3}$$

is a rational general solution of the given differential equation.

Remark 2.1.7. Above results have shown that the rational general solution of the autonomous AODE (2.1) can not be found outside such autonomous AODEs of genus zero. This situation is not true in the case of determining rational general solutions of a first-order non-autonomous AODE, see [57, Section 3]. There are slight differences between a *strong rational general solution* in [57, Definition 3.3] and a rational general solution in the sense of Ritt [43], see also Definition 2.1.1. If a first-order AODE is *parametrizable*, see [57, Definition 3.2], i.e. the corresponding algebraic curve has genus zero, then such general solutions are coincided. Hence, the rational general solutions considered in [7, 33] are strong. In Section 4.1, we recall and tailor the approach of [7, 57] for finding liouvillian solutions of first-order AODEs of genus zero.

In this part, we present main idea of the approach in Aroca et al. [1] for finding a non-constant algebraic solution ξ of the first-order autonomous AODE (2.1) based on the method of [14]. In stead of using global parametrizations, they process with local parametrizations respect to a given tuple, see [59]. From [1, Lemma 2.4], if $G(x, y)$ is an irreducible annihilating polynomial of ξ, then all root $y = y(x)$ of the algebraic equation $G(x, y) = 0$ are solutions of the differential equation. By abuse of notation, $G(x, y) = 0$ is sometimes called a solution. The following lemma gives a form of an algebraic general solution.

Lemma 2.1.8. *([1, Lemma 3.1]) Let $G(x, y) = 0$ be an algebraic solution of the AODE $F(y, y') = 0$ (2.1). Then $G(x+c, y) = 0$ is an algebraic general solution of such AODE (2.1), where c is an arbitrary constant.*

From [1], if the AODE (2.1) has a non-constant algebraic solution, then all its solutions are algebraic. Lemma 2.1.9 shows that the genus of the AODE (2.1) is equal to the genus of its algebraic solutions.

Lemma 2.1.9. *([1, Lemma 3.5]) Assume that $G(x, y) = 0$ is an algebraic solution of the first-order AODE (2.1). Then the genus of $G(x, y) = 0$ is equal to the genus of the corresponding algebraic curve $F(y, w) = 0$.*

Proof. Let η satisfy $G(x, \eta) = 0$. Then η is transcendental over $\overline{\mathbb{Q}}$. Then $\overline{\mathbb{Q}}(x, \eta)$ and $\overline{\mathbb{Q}}(\eta, \eta')$ are the associated algebraic function fields of $G(x, \eta) = 0$ and $F(y, w) = 0$, respectively. Therefore, we only need to prove the two function fields are equal. From Theorem 2.1.11, we have

$$[\overline{\mathbb{Q}}(x, \eta) : \overline{\mathbb{Q}}(\eta)] = [\overline{\mathbb{Q}}(\eta, \eta') : \overline{\mathbb{Q}}(\eta)].$$

Since $G(x, \eta) = 0$, it follows that

$$\eta' = -\frac{\partial G}{\partial x}(x, \eta) / \frac{\partial G}{\partial y}(x, \eta).$$

This yields $\eta' \in \overline{\mathbb{Q}}(x, \eta)$. Hence

$$\overline{\mathbb{Q}}(x, \eta) = \overline{\mathbb{Q}}(\eta, \eta'). \tag{2.6}$$

From (2.6), we observe that $G(x, y) = 0$ and $F(y, w) = 0$ are two birational curves. Due to [59, Chapter VI], their genus are equal. □

Remark 2.1.10. If a first-order AODE (2.1) is of genus zero, then its algebraic general solution is also too. In this case, the associated algebraic function field is of genus zero. This aspect is applied for finding algebraic solutions in Section 3.2.

Assume that $G(x, y) = 0$ is a non-constant algebraic solution of (2.1). Then its degree will be bounded by Theorem 2.1.11 and Theorem 2.1.12.

Theorem 2.1.11. *([1, Theorem 3.4]) Let $G(x, y) \in \mathbb{Q}[x, y]$ be an irreducible polynomial and $G(x, y) = 0$ is an algebraic solution of the first-order AODE (2.1). Then we have*

$$\deg_x G = \deg_w F.$$

Theorem 2.1.12. *([1, Theorem 3.8]) Assume that $G(x, y) = 0$ is a non-constant algebraic solution of the AODE (2.1). Then we have*

$$\deg_y G \leq \deg_y F + \deg_w F.$$

Remark 2.1.13. From above results, in next steps, they build a formal power series solution ξ in N terms, for instance see [1, Algorithm 4.3], generated from a point in the corresponding algebraic curve (local parametrization). By [1, Algorithm 4.4], if ξ is an algebraic solution of (2.1) then its irreducible annihilating polynomial $G(x, y)$ must satisfy Theorem 2.1.11 and Theorem 2.1.12. Finally, the implicit form $G(x, y) = 0$ can be determined by solving a linear system. Since there is a bound of $G(x, y)$ with respect to a certain AODE (2.1), then [1, Algorithm 4.4] terminates after finite steps. If this algorithm can not return an algebraic solution, then the AODE (2.1) has no algebraic general solution.

We give some details for a special case of algebraic solutions, i.e. *radical solutions*.

Definition 2.1.14. ([48, Definition 2.1]) Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. A *radical tower* over $\mathbb{K}(x)$ is a tower of field extension

$$\mathbb{K}(x) = E_0 \subseteq E_1 \subseteq E_2 \subseteq \ldots \subseteq E_m$$

such that for all $i \in \{1, \ldots, m\}$,

$$E_i = E_{i-1}(t_i) = E_0(t_1, t_2, \ldots, t_i) \text{ with } t_i^{n_i} = \alpha_i \in E_{i-1}, n_i \in \mathbb{N}.$$

A field E is called a *radical extension field* of $\mathbb{K}(x)$ if there is a radical tower over $\mathbb{K}(x)$ with E as its last element.

Definition 2.1.15. Let E be a radical extension field of $\mathbb{K}(x)$. An element $y(x) \in E$ is called a *radical function* over $\mathbb{K}(x)$ (briefly, radical function). If such $y(x)$ is a general solution of a first-order AODE (2.1) then we call it a *radical general solution*.

Remark 2.1.16. If $y(x)$ is a radical solution of a first-order AODE (2.1), then it is also an algebraic solution. From Lemma 2.1.8, $y(x + c)$ is a *radical general solution*. Since $\mathbb{K}(x)$ is a radical tower over its self then a rational function in $\mathbb{K}(x)$ can be seen as a radical one. This follows a rational general solution of the AODE (2.1) is also a radical general solution. Moreover, there is a procedure (i.e. [18, PROCEDURE 1]) for finding radical solutions of the AODE (2.1) which extends the method of [14] with respect to *radical parametrizations* (i.e. similar to rational parametrizations but each of these two components are in E), see Example 2.4.11 for an illustration.

2.2 Rational liouvillian solutions

In this section, we define rational liouvillian solutions of first-order autonomous AODEs and show how they differ from the others. First, we recall the result in [56] which has an important role on our method.

Proposition 2.2.1. *([56, Proposition 3.1]) Let $\mathbb{C}$ be an algebraically closed field of characteristic zero (a complex field, in this dissertation) with the standard derivation and let $R(X) \in \mathbb{C}(X)$ be a nonzero element. The differential equation*

$$y' = R(y) \tag{2.7}$$

has a non-constant solution y which is liouvillian over $\mathbb{C}$ if and only if there is an element $z \in \mathbb{C}(y)$ such that $\frac{1}{R(y)}$ is of the form $\frac{\partial z}{\partial y}$ or $\frac{\frac{\partial z}{\partial y}}{az}$ for some $a \in \mathbb{C}\backslash 0$.

Proof. Let y be a non-constant liouvillian solution of differential equation (2.7). Then y is transcendental over $\mathbb{C}$ and $y' \in \mathbb{C}(y)$. That means $\mathbb{C}(y, y')$ is a differential field. For $z \in \mathbb{C}(y)$, we have

$$z' = y' \frac{\partial z}{\partial y}. \tag{2.8}$$

If $z' = 0$, then $\frac{\partial z}{\partial y} = 0$ since the field of constants of $\mathbb{C}(y)$ with the derivation $\frac{\partial}{\partial y}$ equals $\mathbb{C}$, we obtain that $z \in \mathbb{C}$. Since y is liouvillian, the differential field $C(y)$ is contained in some liouvillian extension field of $\mathbb{C}$. By [56, Theorem 2.2], there exists an element $z \in \mathbb{C}(y) \backslash \mathbb{C}$ such that either $z' = 1$ or $z' = az$ for some non-zero $a \in \mathbb{C}$. It follows immediately from the equation (2.7) that $1/R(y)$ has the desired form. In the inverse, let $y' = R(y)$ and $1/R(y)$ be equal to $\frac{\partial z}{\partial y}$ or $\frac{\frac{\partial z}{\partial y}}{az}$ for some $z \in \mathbb{C}(y)$ and for some non-zero $a \in \mathbb{C}$. From (2.8), we obtain $z' = 1$ or $z' = az$. From the fact that $\mathbb{C}(z) \subseteq \mathbb{C}(y)$, it follows that y is algebraic over the liouvillian extension $C(z)$ of $\mathbb{C}$ and thus $\mathbb{C}(y)$ is a liouvillian extension of $\mathbb{C}$. □

Proposition 2.2.1 suggests the following definition.

Definition 2.2.2. [35, Definition 2.7] Let E be a liouvillian extension of $\mathbb{C}$, and $t \in E \backslash \mathbb{C}$. The element t is called a *rational liouvillian element* over $\mathbb{C}$ if $t' \in \mathbb{C}(t)$.

In other words, a rational liouvillian element is a non-constant liouvillian solution of a differential equation (2.7). It is clear that if t is a rational liouvillian element over $\mathbb{C}$, then $\mathbb{C}(t)$ is a differential field.

Definition 2.2.3. [35, Definition 2.8] Let $F(y, y') = 0$ be a first-order autonomous AODE. A solution $y = r(t)$ of the differential equation $F(y, y') = 0$ is called a *rational liouvillian solution* over $\mathbb{C}$ if it is of the form

$$r(t) = \frac{a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0}{b_m t^m + b_{m-1} t^{m-1} + \cdots + b_1 t + b_0},$$

where $m, n \in \mathbb{N}$, $a_i, b_j \in \mathbb{C}$ and t is a rational liouvillian element over $\mathbb{C}$.

If m and n are not both zero, then a rational liouvillian solution is non-constant. Since $\mathbb{C}$ is algebraically closed and t is a rational liouvillian element over $\mathbb{C}$, t must be transcendental over $\mathbb{C}$. If $t' \in \mathbb{C}$ (in particular, $t' = 1$), the differential field $\mathbb{C}(t)$ is exactly the same as the field of rational functions $\mathbb{C}(x)$. In this case, a rational liouvillian solution is a rational solution. Otherwise, the rational liouvillian solutions are really different from the set of rational solutions in [14], radical solutions in [18], fractional rational solutions in [3], and algebraic solutions in [1]. In fact, we make it clearly by considering two simple examples in [2].

Example 2.2.4. The differential equation

$$y' - y = 0$$

admits $\exp x$ as a transcendental solution which is not rational. Setting $t = \exp x$, we have $t' = t$. This implies $\frac{t'}{t} = 1 \in \mathbb{C}$. Hence, $y = r(t) = t$ is a rational liouvillian solution over $\mathbb{C}$ of the given equation.

Example 2.2.5. The differential equation

$$y' - y^2 - 1 = 0$$

admits $y = \tan x$ as a transcendental solution which is not rational. Setting $t = \exp(ix)$, this follows $\frac{t'}{t} = i \in \mathbb{C}$. Since

$$\tan x = \frac{i(1 - \exp(2ix))}{1 + \exp(2ix)},$$

then our solution can be written as

$$y = \tan x = r(t) = \frac{i(1 - t^2)}{1 + t^2}$$

which is a rational liouvillian solution over $\mathbb{C}$ of the given equation.

Remark 2.2.6. Grasegger also provides two examples about non-radical solutions, see [18, Example 3 and Example 4], which are rational liouvillian solutions in these cases.

It is not hard to see that a rational liouvillian element over $\mathbb{C}$ or a rational liouvillian solution of an autonomous AODE of order one is also a liouvillian solution. However, the inverse is not true, for instance see Example 2.4.9. Here, Example 2.2.7 shows that a liouvillian solution of an autonomous AODE of order one may not be a rational liouvillian element over $\mathbb{C}$.

Example 2.2.7. [35, Example 2.3] The differential equation

$$y^2 y'^2 - y^2 + 1 = 0$$

has a solution $y(x) = \sqrt{x^2 + 1}$. Since $y(x)$ is algebraic over $\mathbb{C}(x)$, then it is liouvillian over $\mathbb{C}$. Set $t = y(x)$, we prove that $t' \notin \mathbb{C}(t)$. In fact, if $t' = \frac{x}{\sqrt{x^2+1}} \in \mathbb{C}(t)$, then $x \in \mathbb{C}(\sqrt{x^2 + 1})$. So $\mathbb{C}(x)$ would coincide with $\mathbb{C}(\sqrt{x^2 + 1})$. From Lüroth theorem in [58, Section 63], we have

$$\sqrt{x^2 + 1} = \frac{ax + b}{cx + d}, \text{ with } a, b, c, d \in \mathbb{C}, ad - bc \neq 0.$$

Therefore,

$$(cx + d)^2(x^2 + 1) = (ax + b)^2.$$

This leads to $a = b = c = d = 0$ which is a contradiction. Hence $t = \sqrt{x^2 + 1}$ is not a rational liouvillian element over $\mathbb{C}$.

2.3 Main results

This section gives necessary and sufficient conditions for a first-order autonomous AODE $F(y, y') = 0$ (2.1) having a non-constant rational liouvillian solution.

Lemma 2.3.1. *[35, Lemma 3.1] If the autonomous AODE $F(y, y') = 0$ (2.1) has a non-constant rational liouvillian solution then the corresponding algebraic curve $F(y, w) = 0$ is rational.*

Proof. If $y = r(t)$ is a non-constant rational liouvillian solution of $F(y, y') = 0$, then

$$y' = r'(t) = \frac{dr}{dt}t' \in \mathbb{C}(t).$$

Hence $(r(t), r'(t))$ is a rational parametrization of the algebraic curve $F(y, w) = 0$. Therefore, the algebraic curve $F(y, w) = 0$ is rational. Note that such a parametrization may be not a proper parametrization, for instance, see Remark 2.3.3. □

Remark 2.3.2. From Lemma 2.3.1, if $F(y, w) = 0$ is not rational then the first-order AODE $F(y, y') = 0$ (2.1) has no non-constant rational liouvillian solution.

Remark 2.3.3. If $t' = 1$ then $\mathbb{C}(t)$ coincides with $\mathbb{C}(x)$. This leads to $(r(t), r'(t))$ is a proper paramatrization, see Theorem 2.1.3 and Theorem 1.5.7. Otherwise, since t is transcendental over $\mathbb{C}$, from [4, Theorem 3.2.2] we may take $t' = at + b$ with $a \neq 0$ and choose $r(t) = (at + b)^2$, then $r'(t) = 2(at + b)t' = 2(at + b)^2$. It follows that

$$\mathbb{C}(r(t), r'(t)) = \mathbb{C}((at + b)^2, 2(at + b)^2) = \mathbb{C}((at + b)^2) \neq \mathbb{C}(t).$$

From Theorem 1.5.7, $(r(t), r'(t))$ is not a proper parametrization.

Lemma 2.3.4 gives a sufficient condition for a first-order autonomous $F(y, y') = 0$ having a non-constant rational liouvillian solution.

Lemma 2.3.4. *[35, Lemma 3.2] Let $(r_1(t), s_1(t))$ be a proper rational parametrization of $F(y, w) = 0$. If the AODE $F(y, y') = 0$ has a non-constant rational liouvillian solution then $\dfrac{\frac{dr_1}{dt}}{s_1(t)}$ must be of the form $\dfrac{dz}{dt}$ or $\dfrac{\frac{dz}{dt}}{az}$, where $z \in \mathbb{C}(t)$ and $a \in \mathbb{C}\backslash 0$.*

Proof. Suppose that $r(t)$ is a non-constant rational liouvillian solution of $F(y, y') = 0$. By Lemma 2.3.1, $(r(t), r'(t))$ is a rational parametrization of $F(y, w) = 0$. From Lemma 1.5.8, there is a function $\varphi(t) \in \mathbb{C}(t)$ such that $(r_1(\varphi(t)), s_1(\varphi(t))) = (r(t), r'(t))$. Thus,

$$\frac{d\varphi}{dt}t' = \varphi'(t) = \frac{s_1(\varphi(t))}{\frac{dr_1}{dt}(\varphi(t))} \in \mathbb{C}(\varphi(t)).$$

Since $r(t)$ is a rational liouvillian solution, then t is a rational liouvillian element over $\mathbb{C}$. Hence $\varphi(t)$ is liouvillian over $\mathbb{C}$. From the above formula, we have $\varphi'(t) \in \mathbb{C}(\varphi(t))$. This follows that $\varphi(t)$ is a rational liouvillian element over $\mathbb{C}$. From Proposition 2.2.1, $\frac{\frac{dr_1}{dt}}{s_1(t)}$ must be of the form $\frac{dz}{dt}$ or $\frac{\frac{dz}{dt}}{az}$, where $z \in \mathbb{C}(t)$ and $a \in \mathbb{C}\backslash 0$. □

Lemma 2.3.5 shows the form of rational liouvillian solutions via an indeterminate x, where $x' = 1$.

Lemma 2.3.5. *[35, Lemma 3.3] Let $(r(t), s(t))$ be a proper parametrization of the rational algebraic curve $F(y, w) = 0$. Setting $h(t) = \frac{\frac{dr}{dt}}{s(t)}$, we have two cases:*

1. If there is an element $z(t) \in \mathbb{C}(t)$ such that $h(t) = \frac{dz}{dt}$, then by setting $z(t) = x$, we obtain $r(t)$ is a rational liouvillian solution of $F(y, y') = 0$.

2. If there is an element $z(t) \in \mathbb{C}(t)$ such that $h(t) = \frac{\frac{dz}{dt}}{az}$ for some $a \in \mathbb{C}\backslash 0$, then by setting $z(t) = \exp ax$, we obtain $r(t)$ is a rational liouvillian solution of $F(y, y') = 0$.

Proof. 1. If there is an element $z(t) \in \mathbb{C}(t)$ such that $h(t) = \frac{dz}{dt}$, then setting $z(t) = x$, we have $z'(t) = \frac{dz}{dt}t' = x' = 1$. This implies that $t' = \frac{1}{h(t)} \in \mathbb{C}(t)$ and $r'(t) = s(t)$. Hence $y = r(t)$ is a solution of $F(y, y') = 0$. Since $z(t) = x$, then t is algebraic over $\mathbb{C}(x)$, this yields t is liouvillian over $\mathbb{C}$. Since $t' = \frac{1}{h(t)} \in \mathbb{C}(t)$, then t is a rational liouvillian element over $\mathbb{C}$. Hence, such $y = r(t)$ is a rational liouvillian solution of $F(y, y') = 0$.

2. If there is an element $z(t) \in \mathbb{C}(t)$ such that $h(t) = \frac{\frac{dz}{dt}}{az}$ for some non-zero $a \in \mathbb{C}$, then by setting $z(t) = \exp ax$, we have $z'(t) = \frac{dz}{dt}t' = a \exp ax = az$. This follows that $t' = \frac{1}{h(t)} \in \mathbb{C}(t)$ and $r'(t) = s(t)$. Hence $y = r(t)$ is a solution of $F(y, y') = 0$. Since $\exp ax$ is liouvillian over $\mathbb{C}$ and t is algebraic over $\mathbb{C}(\exp ax)$, then t is liouvillian over $\mathbb{C}$. From above, since $t' \in \mathbb{C}(t)$, then t is a rational liouvillian element over $\mathbb{C}$. Therefore, such $y = r(t)$ is a rational liouvillian solution of $F(y, y') = 0$. □

Lemma 2.3.6 reveals the existence of a rational liouvillian element t is independent of the choice of a proper parametrization of an algebraic curve.

Lemma 2.3.6. *[35, Lemma 3.4] Let $F(y,w) = 0$ be a rational algebraic curve over $\mathbb{C}$. Suppose that $(r_1(t), s_1(t))$ and $(r_2(t), s_2(t))$ are two different proper rational parametrizations of the curve $F(y,w) = 0$. Then the two following differential equations*

$$t' = \frac{s_1(t)}{\frac{dr_1(t)}{dt}} \quad \text{and} \quad t' = \frac{s_2(t)}{\frac{dr_2(t)}{dt}}$$

have the same liouvillian solvability.

Proof. Assume that $(r_1(t), s_1(t))$ and $(r_2(t), s_2(t))$ are two different proper rational parametrizations of $F(y,w) = 0$, by Lemma 1.5.8, there exists a rational function $\varphi(t) = \frac{at+b}{ct+d}$, with $a, b, c, d \in \mathbb{C}, ad-bc \neq 0$ such that $(r_1(\varphi(t)), s_1(\varphi(t))) = (r_2(t), s_2(t))$. Assume that the equation $t' = \frac{s_2(t)}{\frac{dr_2(t)}{dt}}$ has a liouvillian solution t over $\mathbb{C}$, then there exists a liouvillian extension field E of $\mathbb{C}$ containing t. Then

$$t' = \frac{s_2(t)}{\frac{dr_2(t)}{dt}} = \frac{s_1(\varphi(t))}{\frac{dr_1}{dt}(\varphi(t))\frac{d\varphi(t)}{dt}}.$$

It follows $s = \varphi(t)$ is a solution of the equation $s' = \frac{s_1(s)}{\frac{dr_1(s)}{ds}}$. Since $t \in E$ and $\varphi(t)$ is a rational function over $\mathbb{C}$, then $\varphi(t) \in E$. Hence, $\varphi(t)$ is a liouvillian solution. □

The results from Lemma 2.3.1 to Lemma 2.3.6 and Lemma 1.5.6 lead to the following main theorem.

Theorem 2.3.7. *[35, Theorem 3.1] A first-order autonomous AODE $F(y,y') = 0$ has a non-constant rational liouvillian solution if and only if the algebraic curve $F(y,w) = 0$ is rational and for every proper parametrization $(r(t), s(t))$, there exists $z(t) \in \mathbb{C}(t)$ such that $\frac{\frac{dr}{dt}}{s(t)}$ is of the form $\frac{dz}{dt}$ or $\frac{\frac{dz}{dt}}{az}$ for some non-zero $a \in \mathbb{C}$. In the first case, let $z(t) = x$, and in the second case, let $z(t) = \exp ax$, where $x' = 1$. Then $r(t)$ is a rational liouvillian solution of $F(y,y') = 0$.*

Remark 2.3.8. Theorem 2.3.7 shows that the AODE (2.1) does not have a rational liouvillian solution if its corresponding algebraic curve $F(y,w) = 0$ is not rational. If $F(y,w) = 0$ is of genus zero, then it has a proper parametrization $(r(t), s(t))$. Suppose that $r(t)$ is a rational liouvillian solution of the AODE (2.1), moreover, if there is an expression $t = g(x)$ respect to one of the two above cases $z(t) = x$ and $z(t) = \exp ax$, then $r(g(x))$ is a rational liouvillian solution of $F(y,y') = 0$. In these cases, $r(g(x+c))$ is a *rational liouvillian general solution.*

2.4 An algorithm and examples

In general, we can decide if $h(t) \in \mathbb{Q}(t)$ is one of the two forms described in Theorem 2.3.7 as it was shown by Risch [41, Main Theorem]. Let $F(y, w) \in \mathbb{Q}[y, w]$ be the defining polynomial of an algebraic curve, we provide an algorithm for deciding when $F(y, y') = 0$ has a rational liouvillian solution.

Algorithm `RatLiouSol`

Input: An algebraic curve $F(y, w) = 0$.

Output: A rational liouvillian general solution of $F(y, y') = 0$ if any.

---

1. If the algebraic curve $F(y, w) = 0$ is not rational, then **return** "$F(y, y') = 0$ does not have a rational liouvillian solution". Else,

2. Compute a proper parametrization $(r(t), s(t))$ of $F(y, w) = 0$ and set $h(t) = \dfrac{\frac{dr}{dt}}{s(t)}$.

3. If $h(t)$ is not satisfied the two cases of Theorem 2.3.7, then **return** "$F(y, y') = 0$ has no rational liouvillian solution". Else,

4. If $h(t) = \dfrac{dz}{dt}$ where $z(t) \in \mathbb{C}(t)$, then setting $z(t) = x$. There are some cases.

   (a) If $h(t) = \dfrac{1}{a} \in \mathbb{C}$, then $z(t) = \dfrac{t}{a} = x$. So $t = g(x) = ax$, and $y = r(ax)$ is a rational solution. It also gets $r(a(x + c))$ is a rational general solution.

   (b) If $h(t) = \dfrac{1}{a(t-b)^2}$, then $z(t) = \dfrac{-1}{a(t-b)} = x$, so $t = g(x) = b - \dfrac{1}{ax}$. Hence $y = r\left(b - \dfrac{1}{ax}\right)$ is a rational solution. It also gets $r\left(b - \dfrac{1}{a(x+c)}\right)$ is a rational general solution.

   (c) If $t = g(x)$ and both cases (a) and (b) do not occur, then $F(y, y') = 0$ has a radical solution $r(g(x))$. In this case, $r(g(x + c))$ is a radical general solution.

   (d) If there is not an explicit function $g(x)$ such that $t = g(x)$, then $r(t)$ is a rational liouvillian solution which is not a radical solution.

5. If $h(t) = \dfrac{\frac{dz}{dt}}{az}$ with $z(t) \in \mathbb{C}(t)$, then we set $z(t) = \exp ax$. Assume $t = g(x)$, then we get $r(g(x))$ is a rational liouvillian solution of $F(y, y') = 0$ which is not algebraic. In this case, $r(g(x + c))$ is a rational liouvillian general solution.

Remark 2.4.1. MAPLE can help us to compute a proper rational parametrization of $F(y, w) = 0$ and even $h(t)$ if the curve is rational, by following commands.

```
> with(algcurves):
> F:=F(y,w);
> g:=genus(F,y,w);
> P:=parametrization(F,y,w,t); # (only exists when g=0)
> h:=simplify(diff(P[1],t)/P[2]);
```

Although Algorithm `RatLiouSol` is based on rational parametrizations of the corresponding algebraic curves, the solutions are not just rational solutions. The following examples make this clearly.

Example 2.4.2. The differential equation

$$F(y, y') = 229 - 144y + 16yy'^2 + 16y^4 - 128y^2 + 4yy'^3 + 4y^3 - 4y^3y'^2 - y^2y'^2 + 6y'^2 + y'^3 + y'^4 = 0$$

has the corresponding algebraic curve

$$F(y, w) = 229 - 144y + 16yw^2 + 16y^4 - 128y^2 + 4yw^3 + 4y^3 - 4y^3w^2 - y^2w^2 + 6w^2 + w^3 + w^4 = 0$$

which has a proper parametrization

$$(r(t), s(t)) = \left(\frac{t^3 + t^4 + 1}{t^2}, \frac{t^3 + 2t^4 - 2}{t}\right).$$

Compute

$$h(t) = \frac{\frac{dr}{dt}}{s(t)} = \frac{1}{t^2}.$$

Hence, case 4.(b) occurs. It follows a rational general solution of the given equation is

$$r\left(\frac{-1}{x + c}\right) = \frac{(x + c)^4 - (x + c) + 1}{(x + c)^2}.$$

Example 2.4.3. Consider the differential equation

$$y^5 - y'^2 = 0.$$

Its corresponding algebraic curve $y^5 - w^2 = 0$ obtains a rational parametrization

$$(r(t), s(t)) = (t^2, t^5).$$

From that,

$$h(t) = \frac{2}{t^4} = \frac{dz}{dt}, \text{ with } z(t) = \frac{-2}{3t^3} = x.$$

This follows $t = \sqrt[3]{\frac{-2}{3x}}$. Hence, a rational liouvillian solution of the given equation is

$$r\left(\sqrt[3]{\frac{-2}{3x}}\right) = \sqrt[3]{\frac{4}{9x^2}}.$$

We find that a rational liouvillian general solution is

$$r\left(\sqrt[3]{\frac{-2}{3(x+c)}}\right) = \sqrt[3]{\frac{4}{9(x+c)^2}}.$$

Remark 2.4.4. In the above example, the rational liouvillian solution $y = \sqrt[3]{\frac{4}{9x^2}}$ is a *fractional rational solution* in [3], a radical solution in [18]. A fractional rational solution is a special case of rational liouvillian solutions. The algebraic curve $y^5 - w^2 = 0$ has a parametrization $\left(\sqrt[3]{\frac{4}{9t^2}}, \sqrt[3]{\frac{-32}{243t^5}}\right)$ which is not a rational parametrization. In [18], such a parametrization $\left(\sqrt[3]{\frac{4}{9t^2}}, \sqrt[3]{\frac{-32}{243t^5}}\right)$ is called radical parametrization.

Example 2.4.5. Consider the differential equation

$$5y'^6 - 3y'^2 - 6y + 7 = 0.$$

Its corresponding algebraic curve

$$5w^6 - 3w^2 - 6y + 7 = 0$$

has a proper rational parametrization

$$(r(t), s(t)) = \left(\frac{5t^6 - 3t^2 + 7}{6}, t\right).$$

We obtain $h(t) = 5t^4 - 1$. It follows

$$z(t) = t^5 - t = x.$$

From Theorem 2.3.7, $r(t)$ is a rational liouvillian solution. However, we can not express t in terms of x by radicals. Therefore, $r(t)$ is not a radical solution of the given equation.

Example 2.4.6. Consider the differential equation

$$y^2 + y'^2 = 1.$$

Its corresponding algebraic curve $y^2 + w^2 = 1$ has a proper rational parametrization

$$(r(t), s(t)) = \left(\frac{2t}{t^2+1}, \frac{-t^2+1}{t^2+1}\right).$$

Compute

$$h(t) = \frac{2}{t^2+1} = \frac{\frac{dz}{dt}}{iz},$$

where

$$z = \frac{t-i}{t+i} \in \mathbb{C}(t), i^2 = -1.$$

By setting

$$z = \frac{t-i}{t+i} = \exp ix,$$

then

$$t' = \frac{t^2+1}{2} \in \mathbb{C}(t).$$

Hence, a rational liouvillian solution which is not algebraic of the given equation is

$$r(t) = r\left(\frac{-i\exp ix - i}{\exp ix - 1}\right) = \frac{i}{2}(\exp ix - \exp(-ix)).$$

Example 2.4.7. The differential equation

$$F(y, y') = y'^5 + y^2y'^3 - 3y^2y'^2 + 3y^2y' - y^2 = 0$$

has no rational liouvillian solutions. In fact, the corresponding algebraic curve

$$F(y, w) = w^5 + y^2w^3 - 3y^2w^2 + 3y^2w - y^2 = 0$$

has a proper rational parametrization

$$(r(t), s(t)) = \left(\frac{t^5}{t^2+1}, \frac{t^2}{t^2+1}\right).$$

Compute

$$h(t) = \frac{\frac{dr}{dt}}{s(t)} = 3t^2 + 2 - \frac{2}{t^2+1} = \frac{dz_1}{dt} - \frac{\frac{dz_2}{dt}}{iz_2},$$

where

$$z_1(t) = t^3 + 2t \quad \text{and} \quad z_2(t) = \frac{t-i}{t+i}.$$

Case (3.) occurs, hence, the given equation has no rational liouvillian solution.

Remark 2.4.8. Maple may help us to compute a proper parametrization of a rational curve $F(y, w) = 0$. However, in some cases we do not use exactly the parametrization obtained by Maple 2022. For instance, in Example 2.4.2, by Maple, we get

$$(r(t), s(t)) = \left(\frac{t^4 + 2t^3 + 16}{4t^2}, \frac{t^4 + t^3 - 16}{4t}\right).$$

Setting $\varphi(t) = 2t$, then we have

$$(r(\varphi(t)), s(\varphi(t))) = \left(\frac{t^3 + t^4 + 1}{t^2}, \frac{t^3 + 2t^4 - 2}{t}\right).$$

In Example 2.4.7, by MAPLE, we get a proper parametrization $(r(t), s(t))$ is

$$\left(-\frac{1419857t^5 - 417605t^4 + 49130t^3 - 2890t^2 + 85t - 1}{19520t^5 - 17328t^4 + 5804t^3 - 905t^2 + 66t - 2}, \frac{289t^2 - 34t + 1}{305t^2 - 42t + 2}\right).$$

Setting $\varphi(t) = \dfrac{t+1}{4t+17}$, then we obtain

$$(r(\varphi(t)), s(\varphi(t))) = \left(\frac{t^5}{t^2+1}, \frac{t^2}{t^2+1}\right).$$

In both cases, by Lemma 1.5.8, $(r(\varphi(t)), s(\varphi(t)))$ are proper parametrizations. Since Algorithm `RatLiouSol` is independent of the choice of proper parametrizations of an algebraic curve, then we select the one which is suitable for computation.

Example 2.4.9. In Example 2.2.7, the liouvillian solution $y(x) = \sqrt{x^2+1}$ is not a rational liouvilian element over $\mathbb{C}$. However, we will see that $y(x) = \sqrt{x^2+1}$ can be expressed as a rational function of a rational liouvilian element over $\mathbb{C}$. In fact, we will show that $y(x)$ is also a rational liouvillian solution of the given autonomous AODE

$$F(y, y') = y^2 y'^2 - y^2 + 1 = 0.$$

The corresponding algebraic curve has a proper parametrization

$$(r(t), s(t)) = \left(\frac{2t^2 - 2t + 1}{2t - 1}, \frac{2t(t-1)}{2t^2 - 2t + 1}\right).$$

Then $h(t) = 1 + \dfrac{1}{(2t-1)^2}$. It turns out $z(t) = t - \dfrac{1}{2(2t-1)}$. We set

$$t - \frac{1}{2(2t-1)} = x + \frac{1}{2},$$

and choose

$$t = g(x + \frac{1}{2}) = \frac{x + 1 + \sqrt{x^2+1}}{2}.$$

This leads to a rational liouvillian solution

$$r(t) = r(g(x + \frac{1}{2})) = \sqrt{x^2+1} = y(x).$$

If we set

$$z(t) = t - \frac{1}{2(2t-1)} = x + c + \frac{1}{2},$$

then the liouvillian rational general solution of the differential equation is

$$r(g(x+c+\frac{1}{2})) = \sqrt{(x+c)^2+1},$$

which is also a radical general solution. We note that

$$t = \frac{x+1+\sqrt{x^2+1}}{2}$$

is a rational liouvillian element over $\mathbb{C}$ because $t' = \dfrac{1}{h(t)} \in \mathbb{C}(t)$.

Remark 2.4.10. If the corresponding algebraic curve $F(y,w) = 0$ of an autonomous AODE $F(y,y') = 0$ has a rational parametrization and Procedure RadSol works, then a radical solution of such the AODE found in this way is a rational liouvillian solution. In all examples in [18] except for Example 6, radical solutions are rational liouvillian solutions. Hence, if we want to show a radical solution which is not a rational liouvillian solution, it must be not found in a rational parametrization. The next example reveals that a first-order autonomous AODE may have a radical solution even its corresponding algebraic curve does not have a rational parametrization.

Example 2.4.11. ([18, Example 6]) The differential equation

$$F(y,y') = -y^3 - 4y^5 + 4y^7 - 2y' - 8y^2y' + 8y^4y' + 8yy'^2 = 0$$

has a radical general solution $y = r(x) = -\dfrac{\sqrt{1+x+c}}{\sqrt{1+(x+c)^2}}$. The corresponding algebraic curve has a radical parametrization

$$\left(-\frac{\sqrt{1+t}}{\sqrt{1+t^2}}, \frac{t^2+2t-1}{2\sqrt{1+t}(t^2+1)^{3/2}}\right).$$

Since $r(x)$ is algebraic over $\mathbb{C}(x)$, then $r(x)$ is liouvillian over $\mathbb{C}$. From Theorem 1.5.9, the corresponding algebraic curve does not have a rational parametrization since its genus is one (checked by MAPLE in Remark 2.4.1). From Theorem 2.3.7, $r(x)$ is not a rational liouvillian solution because there is no a rational liouvillian solution of $F(y,y') = 0$ outside a rational parametrization.

Conclusion

In this chapter, we give the necessary and sufficient conditions for a first-order autonomous AODE $F(y,y') = 0$ having a non-constant rational liouvillian solution. We also present Algorithm `RatLiouSol` for determining such a rational liouvillian solution of $F(y,y') = 0$. However, this algorithm is not complete due to the explicit form of t can not be decided in general. This problem will be investigated in Chapter 3.

Chapter 3

Liouvillian solutions of first-order autonomous AODEs of genus zero

In this chapter, we consider a class of first-order autonomous AODEs (2.1) and study their liouvillian solutions. We prove that a liouvillian solution of a first-order autonomous AODE of genus zero is necessarily a rational liouvillian solution over $\mathbb{C}$. As a consequence, the criterion for the existence of rational liouvillian solutions in Chapter 2 is also applicable to such liouvillian solutions. Based on this, an algorithm has been given to decide whether such differential equation has a liouvillian solution and actually compute it in the affirmative case. This algorithm is arranged in such a way that it classifies the types of solutions such as algebraic solutions and non-algebraic solutions. In the first case, this algorithm, which is different from [1, Algorithm 4.4], provides a way to compute it in the implicit form $G(x, y) = 0$. In the second case, the algorithm finds a convergence function whose Taylor series is the truncated expression of the formal power series solution determined by [13, Algorithm 1]. In addition, we give a class of first-order AODEs of positive genus (see [56]) which does not have a non-constant liouvillian solution. This induces a question (studied in Chapter 4) if there are first-order AODEs of certain positive genera which obtain non-constant liouvillian solutions? The content of this chapter is mainly based on the author's works in [36].

The structure of the chapter is as follows. Section 3.1 presents Sylvester resultant. Section 3.2 gives the necessary and sufficient conditions for a first-order autonomous AODE (2.1) of genus zero having a liouvillian solution, then also classifies such the solutions (if any) respect to algebraic and transcendental cases. An algorithm and its applications are given in Section 3.3.

3.1 Sylvester resultant

We recall a well-known elimination method to find the implicit algebraic relation from the rational parametrizations.

Definition 3.1.1 (Sylvester resultant)**.** ([4, Definition 1.4.1]) Let K be a field and t be an indeterminate over K. Let $A, B \in K[t]\backslash 0$. Write

$$A = a_n t^n + \cdots + a_1 t + a_0$$

and

$$B = b_m t^m + \cdots + b_1 t + b_0$$

where $a_n \neq 0, b_m \neq 0$ and at least one of n or m is nonzero. The Sylvester matrix of A and B is the $(n+m) \times (n+m)$ matrix defined by

$$S(A,B) = \begin{pmatrix} a_n & \cdots & \cdots & \cdots & a_1 & a_0 & & \\ & & & \ddots & & & & \\ & & a_n & \cdots & \cdots & \cdots & a_1 & a_0 \\ b_m & \cdots & b_1 & b_0 & & & & \\ & & \ddots & & & & & \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & b_m & \cdots & b_1 & b_0 \end{pmatrix}$$

where the $A-$rows are repeated m times and $B-$rows are repeated n times. The resultant of A and B is the determinant of $S(A,B)$, denoted by $res(A,B)$.

Let $\mathcal{P}(t) = \left(\dfrac{m(t)}{n(t)}, \dfrac{p(t)}{q(t)}\right)$, where $m(t), n(t), p(t), q(t) \in \mathbb{C}[t]$, and consider the following polynomials

$$G_1^{\mathcal{P}}(s,t) = m(s)n(t) - n(s)m(t), \quad G_2^{\mathcal{P}}(s,t) = p(s)q(t) - q(s)p(t)$$

as well as the polynomials

$$H_1^{\mathcal{P}}(t,x) = x.n(t) - m(t), \quad H_2^{\mathcal{P}}(t,y) = y.q(t) - p(t).$$

The resultant (see in [4, 50]) with respect to t of $H_1^{\mathcal{P}}(t,x)$ and $H_2^{\mathcal{P}}(t,y)$ denoted by

$$res_t(H_1^{\mathcal{P}}(t,x), H_2^{\mathcal{P}}(t,y)).$$

Lemma 3.1.2. *([50, Lemma 4.6]) Let $\mathcal{C}$ be a rational curve defining by $F(x,y)$, and $\mathcal{P}(t)$ be a rational parametrization of $\mathcal{C}$. Then there exists $h \in \mathbb{N}$ such that*

$$res_t(H_1^{\mathcal{P}}(t,x), H_2^{\mathcal{P}}(t,y)) = (F(x,y))^h.$$

Remark 3.1.3. Lemma 3.1.2 helps us to find the defining polynomial of a rational curve via its parametrization. From [50] also,

$$h = \deg_t(\gcd(G_1^{\mathcal{P}}(s,t), G_2^{\mathcal{P}}(s,t))).$$

Moreover, $\mathcal{P}(t)$ is a proper parametrization of $\mathcal{C}$ if and only if $h = 1$.

3.2 Main results

First, we recall the definition of liouvillian solutions.

Definition 3.2.1. A solution η of the differential equation $F(y, y') = 0$ is called a *liouvillian solution* over $\mathbb{C}$ if η belongs to some liouvillian extension of $\mathbb{C}$.

Next, Lemma 3.2.2 shows a relationship between rational liouvillian solutions studied in Chapter 2 and liouvillian solutions of $F(y, y') = 0$ when its corresponding curve $F(y, w) = 0$ has genus zero.

Lemma 3.2.2. *[36, Lemma 3.2] Let $F(y, w) = 0$ be a rational curve. Assume that η is a liouvillian solution of $F(y, y') = 0$ over $\mathbb{C}$ then η is a rational liouvillian solution over $\mathbb{C}$.*

Proof. If η is a non-constant liouvillian solution of $F(y, y') = 0$, then η is transcendental over $\mathbb{C}$. From Lemma 1.5.3, since $F(\eta, \eta') = 0$, then $\mathbb{C}(\eta, \eta')$ is an associated field of algebraic functions of the rational curve $F(y, w) = 0$. Moreover, such the function field $\mathbb{C}(\eta, \eta')$ is of the form $\mathbb{C}(t)$. Therefore, we can write

$$\eta = r(t), \eta' = s(t), \text{ with } r(t), s(t) \in \mathbb{C}(t).$$

Since

$$\eta' = \frac{dr}{dt} t' = s(t),$$

then

$$t' = \frac{1}{h(t)} \in \mathbb{C}(t).$$

Since $\eta = r(t)$ is a liouvillian solution over $\mathbb{C}$ and t is algebraic over $\mathbb{C}(\eta)$, then t belongs to some liouvillian extension of $\mathbb{C}$. Hence, t is a rational liouvillian element over $\mathbb{C}(t)$. That means $\eta = r(t)$ is a rational liouvillian solution. $\square$

Finally, Theorem 2.3.7 and Lemma 3.2.2 motivate Theorem 3.2.3 which provides the following ideas for determining algebraic and non-algebraic liouvillian solutions.

Theorem 3.2.3. *[36, Theorem 3.3] Let $F(y,w)=0$ be a rational curve. Then the first-order autonomous AODE $F(y,y')=0$ has a liouvillian solution over $\mathbb{C}$ if and only if for every proper parametrization $(r(t),s(t))$ of $F(y,w)=0$, there exists $z(t)\in\mathbb{C}(t)$ such that the associated function $h(t)$ is either of the form $\frac{dz}{dt}$ or $\frac{\frac{dz}{dt}}{az}$ for some non-zero $a\in\mathbb{C}$. In the first case, let $z(t)=x$, and in the second case, let $z(t)=\exp(ax)$, then $r(t)$ is a liouvillian solution of $F(y,y')=0$.*

Remark 3.2.4. Theorem 3.2.3 is not true if $F(y,w)=0$ is not a rational curve. For instance, [1, Example 4.5] shows that an AODE, whose corresponding algebraic curve is not rational, has an algebraic general solution which is certainly a liouvillian solution over $\mathbb{C}$.

In Chapter 2, Algorithm `RatLiouSol` gives a way to compute a rational liouvillian solution of $F(y,y')=0$. Unfortunately, this algorithm did not let we know whether all algebraic solutions can be found. Certainly, similar questions arisen in the case of non-algebraic solutions. In the rest of this chapter, we make the two issues clearly. More precisely, with respect to the two cases of Theorem 3.2.3, liouvillian solutions are classified in the case of algebraic solutions and non-algebraic solutions.

In the first case, we prove that $F(y,y')=0$ only has algebraic solutions which is ensured by Theorem 3.2.8. This idea will be illustrated by the following results.

Lemma 3.2.5. *[36, Lemma 3.5] Assume that $F(y,w)=0$ has a proper parametrization $(r(t),s(t))$ such that $h(t)$ is of the form $\frac{dz}{dt}$, $z(t)\in\mathbb{C}(t)$. If $F(y,w)=0$ has another parametrization $(r_1(t),s_1(t))$, then $h_1(t)$ is of the form $\frac{dz_1}{dt}$, $z_1(t)\in\mathbb{C}(t)$.*

Proof. From Lemma 1.5.8, there is a $\varphi(t)\in\mathbb{C}(t)$, such that

$$r_1(t)=r(\varphi(t)) \text{ and } s_1(t)=s(\varphi(t)).$$

Then we have

$$\frac{\frac{dr_1}{dt}}{s_1(t)}=\frac{\frac{dr}{d\varphi}\frac{d\varphi}{dt}}{s(\varphi(t))}=\frac{dz}{d\varphi}\frac{d\varphi}{dt}=\frac{d(z(\varphi(t)))}{dt}.$$

By setting $z_1(t)=z(\varphi(t))\in\mathbb{C}(t)$, we obtain $h_1(t)=\frac{\frac{dr_1}{dt}}{s_1(t)}=\frac{dz_1}{dt}$. □

Next lemma is a special case of Lemma 2.1.9.

Lemma 3.2.6. *Let $F(y, w) = 0$ be a rational curve. If $G(x, y) = 0$ is an algebraic solution of $F(y, y') = 0$, then the genus of $G(x, y) = 0$ is zero.*

Remark 3.2.7. Lemma 3.2.5 shows that the form of the associated function $h(t)$ is independent of the choice of proper parametrizations of the rational curve. This follows that the statement in Theorem 3.2.8 is also independent of the choice of proper parametrizations of the rational curve.

Theorem 3.2.8. *[36, Theorem 3.8] Let $F(y, w) = 0$ be a rational curve. The AODE $F(y, y') = 0$ has an algebraic solution $G(x, y) = 0$ if and only if the associated function $h(t)$ is of the form $\frac{dz}{dt}$, where $z(t) \in \mathbb{C}(t)$. If the solution exists, the defining polynomial $G(x, y)$ can be determined by its parametrization.*

Proof. (Necessary) Since $F(y, w) = 0$ is a rational curve, then it has a proper parametrization $(r(t), s(t))$. From Theorem 3.2.3, if $h(t)$ is of the form $\frac{dz}{dt}$, then $y = r(t)$ is a liouvillian solution of $F(y, y') = 0$, with $z(t) = x$. By setting

$$\mathcal{P}(t) = (z(t), r(t)),$$

then it is a rational parametrization of an algebraic curve

$$G(x, y) = 0$$

whose defining polynomial can be found by using the resultant in Lemma 3.1.2. In this case, $G(x, y) = 0$ is an algebraic solution of $F(y, y') = 0$.

(Sufficient) Assume that $G(x, y) = 0$ is an algebraic solution of $F(y, y') = 0$. From Lemma 3.2.6, the genus of $G(x, y) = 0$ is zero. Therefore, it has a proper parametrization

$$\mathcal{P}(t) = (z(t), r(t)).$$

Since $y = r(t)$ is a liouvillian solution of $F(y, y') = 0$, then $(r(t), r'(t))$ is a rational parametrization of $F(y, w) = 0$. Setting $s(t) = r'(t)$, since

$$x' = \frac{dz}{dt} t' = 1,$$

then

$$h(t) = \frac{\frac{dr}{dt}}{s(t)} = \frac{\frac{dr}{dt}}{r'(t)} = \frac{1}{t'} = \frac{dz}{dt}.$$

The proof is complete. □

By Theorem 3.2.8, if the second case of Theorem 3.2.3 occurs, then the AODE $F(y, y') = 0$ must have non-algebraic solutions. Moreover, the form of these solutions can be determined by the next theorem.

Theorem 3.2.9. *[36, Theorem 3.9] Let $F(y, w) = 0$ be a rational curve. Assume that η is a non-algebraic liouvillian solution of $F(y, y') = 0$. Then there are a non-zero element $a \in \mathbb{C}$ and an irreducible polynomial G such that $G(\exp(ax), \eta) = 0$. In other words, η is algebraic over $\mathbb{C}(\exp(ax))$.*

Proof. From Lemma 3.2.2, if η is a liouvillian solution of $F(y, y') = 0$, we can write such solution η by

$$\eta = \eta(t) \in \mathbb{C}(t).$$

Therefore, the pair

$$(\eta(t), \eta'(t))$$

is a rational parametrization of $F(y, w) = 0$. Assume that $(r(t), s(t))$ be another proper parametrization of $F(y, w) = 0$. Since α is a non-algebraic liouvillian solution, from Theorem 3.2.3, there is a $z(t) \in \mathbb{C}(t)$ and a non-zero $a \in \mathbb{C}$ such that

$$h(t) = \frac{\frac{dz}{dt}}{az}.$$

By Lemma 1.5.8, there is a $\varphi(t) \in \mathbb{C}(t)$ such that

$$(\eta(t), \eta'(t)) = (r(\varphi(t)), s(\varphi(t))).$$

Therefore, we have

$$\frac{1}{t'} = \frac{\frac{d(r(\varphi(t)))}{dt}}{s(\varphi(t))} = \frac{\frac{dr}{d\varphi}\frac{d\varphi}{dt}}{s(\varphi(t))} = \frac{\frac{dz(\varphi)}{d\varphi}\frac{d\varphi}{dt}}{az(\varphi)} = \frac{\frac{d(z(\varphi(t)))}{dt}}{a(z(\varphi(t)))}.$$

This means $z(\varphi(t))' = az(\varphi(t))$. By setting

$$u = z(\varphi(t)) = \exp(ax)$$

and using the resultant in Lemma 3.1.2, we can find the implicit form

$$G(u, \eta) = 0$$

from its parametrization $\mathcal{P}(t) = (z(\varphi(t)), \eta(t))$. □

Remark 3.2.10. The results of Theorem 3.2.3, Theorem 3.2.8, and Theorem 3.2.9 ensure that all liouvillian solutions of $F(y, y') = 0$ can be found via proper rational parametrizations of its corresponding rational curve. In the case of genus zero, a liouvillian solution η of $F(y, y') = 0$ is either an algebraic function over $\mathbb{C}(x)$ or an algebraic function over $\mathbb{C}(\exp(ax))$. Therefore, η is an elementary solution of the given AODE (see [51]). In other words, [51, Corollary 2] coincides with [56, Proposition 3.1].

3.3 An algorithm and applications

Let $F(y, w) \in \mathbb{Q}[y, w]$ be the defining polynomial of a rational algebraic curve, we provide an algorithm for deciding when $F(y, y') = 0$ has a liouvillian solution.

Algorithm `LiouSolAut`

Input: A rational algebraic curve $F(y, w) = 0$.

Output: A liouvillian general solution of $F(y, y') = 0$ if any.

---

1. Compute a proper parametrization $(r(t), s(t))$ of the algebraic curve $F(y, w) = 0$ and the associated function $h(t) = \frac{\frac{dr}{dt}}{s(t)}$.

2. If $h(t) = \frac{dz}{dt}$ with $z(t) \in \mathbb{C}(t)$, then set $z(t) = x$ and $\mathcal{P}(t) = (z(t), r(t))$. Set $G(x, y)$ is the square-free part of

$$res_t(H_1^{\mathcal{P}}(t, x), H_2^{\mathcal{P}}(t, y))$$

   in Lemma 3.1.2, then $G(x, y) = 0$ is an algebraic solution. Hence, an algebraic general solution of the given equation is $G(x + c, y) = 0$.

3. If $h(t) = \frac{\frac{dz}{dt}}{az}$ with $z(t) \in \mathbb{C}(t)$, then set $z(t) = \exp(ax) = u$. Set $\mathcal{P}(t) = (z(t), r(t))$, by processing the same way of the case (2.), we obtain $G(u, y) = 0$ is a non-algebraic liouvillian solution. Then $G(\exp(a(x + c)), y) = 0$ is a liouvillian general solution.

4. Otherwise, the algorithm terminates, and $F(y, y') = 0$ has no liouvillian solution.

Remark 3.3.1. [1, Algorithm 4.4] can compute algebraic solutions of $F(y, y') = 0$ regardless the assumption on the corresponding algebraic curve. The main idea of this algorithm is looking for the formal power series solution $\varphi(x) = \Sigma_i a_i x^i$ with degree bound of i, from that, an algebraic solution $G(x, y(x)) = 0$ can be derived. The idea of [13, Algorithm 1] is based on local parametrizations of the corresponding algebraic curve, then looking for the truncation of a formal power series solution of $F(y, y') = 0$. Clearly, Algorithm `LiouSolAut` differs from those above algorithms. In the special case, when $z(t) = \frac{at + b}{ct + d}$ holds for the case (2.), Algorithm `LiouSolAut` coincides with Algorithm `RatSol` if we are looking for rational solutions.

In next part, we present some examples to illustrate Algorithm `LiouSolAut`. In addition, the algorithm can be applied for solving the differential equations in [13] in the case that their corresponding curves have genus zero. In particular, Example 3.3.5 gives a solution which is the convergence function of the formal power series solution of the differential equation in [13, Example 1].

Example 3.3.2. The differential equation

$$F(y, y') = y'^2 - yy' + y - 2y' + 2 = 0$$

has no liouvillian solution over $\mathbb{C}$.

In fact, the corresponding curve $F(y, w) = 0$ has a proper parametrization

$$(r(t), s(t)) = \left(\frac{t^2 - 2t + 2}{t - 1}, t \right).$$

Compute

$$h(t) = \frac{t-2}{(t-1)^2} = \frac{1}{t-1} - \frac{1}{(t-1)^2} = \frac{\frac{dz_1}{dt}}{z_1} + \frac{dz_2}{dt},$$

where

$$z_1(t) = t - 1 \text{ and } z_2(t) = \frac{1}{t-1}.$$

Since $h(t) = (z_2 + \log z_1)'$ is neither of the form $(\frac{1}{a} \log z)'$ nor $(z)'$, where

$$\frac{d}{dt} =' \text{ and } z \in \mathbb{C}(t),$$

then case (4.) occurs. Therefore, the given equation has no liouvillian solution over $\mathbb{C}$.

Remark 3.3.3. By similar arguments, Example 2.4.7 also has no liouvillian solution over $\mathbb{C}$. Hence, these two examples have no other solutions defined in Definition 1.1.16. Maple may be used for solving these AODEs, however, their general solutions (if any) are not liouvillian over $\mathbb{C}$.

Example 3.3.4. [36, Example 4.3] Find a solution of the differential equation

$$F(y, y') = 5y'^6 - 3y'^2 - 6y = 0.$$

The corresponding algebraic curve $F(y, w) = 0$ has a proper parametrization

$$(r(t), s(t)) = \left(\frac{5t^6 - 3t^2}{6}, t \right).$$

Compute $h(t) = 5t^4 - 1$ and set $z(t) = t^5 - t = x$. Case (2.) occurs. Therefore, this equation has an algebraic solution $G(x, y) = 0$ with a rational parametrization

$$\mathcal{P}(t) = (z(t), r(t)) = \left(t^5 - t, \frac{5t^6 - 3t^2}{6}\right).$$

In this situation the solution $r(t)$ obtained by Algorithm `RatLiouSol` is not a radical solution. A similar case can be found in Example 2.4.5. Here, we compute

$$G_1^{\mathcal{P}}(s, t) = s^5 - t^5 - s + t, \quad G_2^{\mathcal{P}}(s, t) = 6(5(s^6 - t^6) + 3(-s^2 + t^2))$$

and

$$H_1^{\mathcal{P}}(t, x) = x - t^5 + t, \quad H_2^{\mathcal{P}}(t, y) = 6y - 5t^6 + 3t^2.$$

Since

$$\deg_t(\gcd(G_1(s, t), G_2(s, t))) = \deg_t(t - s) = 1,$$

then the given differential equation has an algebraic solution

$$\begin{aligned} G(x, y) &= res_t(H_1^{\mathcal{P}}(t, x), H_2^{\mathcal{P}}(t, y)) \\ &= -3125x^6 - 11250x^4y + 7776y^5 - 10800x^2y^2 - 1728y^3 + 48x^2 + 96y = 0. \end{aligned}$$

Therefore, $G(x + c, y) = 0$ is an algebraic general solution.

Example 3.3.5. ([13, Example 1]) Consider the differential equation

$$F(y, y') = y'^2 - y^3 - y^2 = 0.$$

The corresponding curve $F(y, w) = 0$ has proper parametrization

$$(r(t), s(t)) = (t^2 - 1, t(t^2 - 1)).$$

Compute $h(t) = \dfrac{2}{1 - t^2}$ and set $z(t) = \dfrac{t - 1}{t + 1}$. Case (3.) occurs. By setting

$$u = z(t) = \exp(x),$$

we obtain

$$P(t) = (\frac{t - 1}{t + 1}, t^2 - 1)$$

is a rational parametrization of $G(u, y) = 0$. By similar computation of Example 3.3.4, we get

$$G(u, y) = (u - 1)^2 y - 4u.$$

By replacing $u = \exp(x)$, the given equation has a liouvillian solution

$$G(\exp(x), y) = (1 - \exp(x))^2 y - 4\exp(x) = 0.$$

In this case, $G(\exp(x + c), y) = 0$ is a liouvillian general solution.

Remark 3.3.6. With $c = \ln(3 - 2\sqrt{2})$ in the above general solution, then

$$y = \frac{4(3 - 2\sqrt{2})\exp(x)}{(1 - (3 - 2\sqrt{2})\exp(x))^2}$$

is a solution of the differential equation in Example 3.3.5. Let us denote $T_3(x)$ to be the Taylor series expansion of y at $x = 0$ with degree 3, we obtain

$$T_3(x) = 1 + \sqrt{2}x + \frac{5}{4}x^2 + \frac{2\sqrt{2}}{3}x^3 + 0(x^3).$$

It is clear that $T_3(x)$ is the solution $A(S)$ (with the local parametization at $\mathbf{c} = (1, \sqrt{2}) \in \mathbf{A}_1$) of the differential equation in [13, Example 1].

Next, we consider a general case of Example 3.3.5, the AODE

$$y'^2 = P(y), \tag{3.1}$$

where $P(y) \in \mathbb{C}[y]$ has degree 3. We start off this issue with the following lemma.

Lemma 3.3.7. *[36, Lemma 4.6] If $F(y, w) = 0$ has a proper parametrization $(r(t), s(t))$ and $h(t)$ is of the form $\frac{1}{at^2 + bt + c}$ where $a, b, c \in \mathbb{C}$, then $F(y, y') = 0$ has a liouvillian solution.*

Proof. There are three cases to consider:

1) If $a = b = 0$, then $h(t) = \frac{1}{c}$. By setting $z(t) = \frac{t}{c}$, then $h(t) = \frac{dz}{dt}$.

2) If $a = 0$, then $h(t) = \frac{1}{bt + c}$. By setting $z(t) = bt + c$, then $h(t) = \frac{\frac{dz}{dt}}{bz}$.

3) If $a \neq 0$, then $at^2 + bt + c = a((t + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2})$. Set $\Delta = b^2 - 4ac$.

   (a) If $\Delta = 0$, by setting $z = \frac{-1}{at + b/2}$, then $h(t) = \frac{dz}{dt}$.

   (b) If $\Delta \neq 0$, then $at^2 + bt + c = a(t + \frac{b}{2a} + \frac{\sqrt{\Delta}}{2a})(t + \frac{b}{2a} - \frac{\sqrt{\Delta}}{2a})$.

   By setting $z(t) = \frac{t + \frac{b}{2a} - \frac{\sqrt{\Delta}}{2a}}{t + \frac{b}{2a} + \frac{\sqrt{\Delta}}{2a}}$, then $h(t) = \frac{\frac{dz}{dt}}{\frac{\sqrt{\Delta}}{a^2}z}$.

Therefore, in all of above three cases, the given equation always has a liouvillian solution by Theorem 3.2.3. □

Proposition 3.3.8. *[36, Proposition 4.7 and Remark 4.8] The AODE (3.1) has a liouvillian solution over $\mathbb{C}$ if an only if $P(y) = 0$ has repeated roots.*

Proof. Assume that $P(y) = 0$ has repeated roots, then it can be written as

$$P(y) = a(y - b)^2(y - c).$$

The corresponding curve of (3.1) has a proper parametrization $(r(t), s(t))$

$$\left(\frac{ab^2ct^2 - 2abct + ac + t^2}{(b^2t^2 - 2bt + 1)a}, \frac{t(ab^3t^2 - ab^2ct^2 - 2ab^2t + 2abct + ab - ac - t^2)}{a(b^3t^3 - 3b^2t^2 + 3bt - 1)}\right).$$

Compute

$$h(t) = \frac{-2}{(ab^3 - ab^2c - 1)t^2 + (-2ab^2 + 2abc)t + ab - ac}.$$

From Lemma 3.3.7, the given equation has a liouvillian solution.

If $P(y) = 0$ has no repeated roots, by Proposition 3.3.10, then the differential equation (3.1) has no non-constant liouvillian solution over $\mathbb{C}$. □

Remark 3.3.9. If $P(y) = 0$ has no repeated roots, by Example 1.4.9, the genus of the algebraic curve

$$w^2 = P(y) \tag{3.2}$$

is one and it is not a rational curve by Theorem 1.5.9. Hence, with Proposition 3.3.8, we can say that the liouvillian solvability of the AODE (3.1) is decidable by checking only the genus of the corresponding algebraic curve.

In final part, we consider a class of first-order autonomous AODEs (3.3) which is a generalization of the AODE (3.2). By Example 1.4.9, the genus of the AODE (3.3) may be reached to any positive number respect to the degree of $P(Y)$. We note that since the associated algebraic function field $\mathbb{C}(y, y')$ of the AODE (3.3) (see Section 1.5.1) is of positive genus (see Theorem 1.3.23) then its behavior, therefore, is different from such the function field of genus zero. For the description of a non-rational differential function field, we refer to [23, 31].

Proposition 3.3.10. *([56, Proposition 3.2]) Let $P(X) \in \mathbb{C}(X)$ be a polynomial of degree ≥ 3 with no repeated roots. Then the differential equation*

$$y'^2 = P(y) \tag{3.3}$$

has no non-constant liouvillian solution over $\mathbb{C}$.

Proof. Suppose that there exists a non-constant liouvillian solution y satisfying the differential equation (3.3). Since $\mathbb{C}$ is algebraically closed, such an element y must be transcendental over $\mathbb{C}$. By [56, Theorem 2.2], there is an element $z \in \mathbb{C}(y, y')\backslash\mathbb{C}$ such that either $z' = 1$ or $z' = az$ for non-zero element $a \in \mathbb{C}$. Since $P(y)$ has no repeated roots then it is not a square of an element in $\mathbb{C}[y]$, that means $y' \notin \mathbb{C}(y)$. Hence, y' belongs to a quadratic extension of $\mathbb{C}(y)$. Moreover, from (2.8), we obtain $z \notin \mathbb{C}(y)$. Hence, we can write

$$z = A + By', \text{ where } A, B \in \mathbb{C}(y), B \neq 0. \tag{3.4}$$

We will show that such an element z of the form (3.4) does not exist. In fact, by taking the derivatives to (3.4), we obtain

$$z' = A' + B'y' + By''.$$

Using (2.8) and (3.3), we have

$$z' = y'\frac{\partial A}{\partial y} + P\frac{\partial B}{\partial y} + \frac{B}{2}\frac{\partial P}{\partial y}.$$

If there is an element z of the form (3.4) such that $z' = az$ then by comparing coefficients of the above equation, we obtain

$$\frac{\partial A}{\partial y} = aB \tag{3.5}$$

and

$$P\frac{\partial B}{\partial y} + \frac{B}{2}\frac{\partial P}{\partial y} = aA. \tag{3.6}$$

Multiplying (3.6) by $2B$ and using (3.5), we obtain

$$\frac{\partial B^2 P}{\partial y} = \frac{\partial A^2}{\partial y}.$$

Hence, there is a non-zero element $c \in \mathbb{C}$ such that

$$B^2 P = A^2 + c. \tag{3.7}$$

Write $A = A_1/A_2$ and $B = B_1/B_2$, where A_1, A_2, B_1, B_2 are polynomials in $\mathbb{C}[y]$ such that A_1, A_2 are relatively prime, B_1, B_2 are relatively prime and A_2, B_2 are monic. Then since P has no square factors, it follows from the equation

$$B_1^2 P A_2^2 = (A_1^2 + cA_2^2)B_2^2$$

that $A_2 = B_2$. From the equation (3.5) and the assumption that A_2 and B_2 are monic, then $A_2 = B_2 = 1$ and $\deg A = 1 + \deg B$. From equation (3.7), then

$$2\deg A - 2 + \deg P = 2\deg A,$$

which follows deg$P = 2$, a contradiction.

If there is an element z of the form (3.4) such that $z' = 1$ then $\dfrac{\partial A}{\partial y} = 0$ and

$$P\frac{\partial B}{\partial y} + \frac{B}{2}\frac{\partial P}{\partial y} = 1.$$

This follows

$$\frac{\partial (B^2P)}{\partial y} = 2B. \tag{3.8}$$

Now we will prove that B does not exist by showing it cannot be a polynomial or has a pole of order ≥ 1. In fact, we obtain a contradiction with (3.3) if $B \in \mathbb{C}[y]$ since

$$\deg \frac{\partial (B^2P)}{\partial y} = 2\deg B + \deg P - 1 \geq 2 + 2\deg B > \deg B.$$

If B has $c \in \mathbb{C}$ is a pole of order $m \geq 1$ then such B can be written as

$$B = f + \sum_{1}^{m} \frac{\alpha_i}{(y-c)^i}$$

where $f \in \mathbb{C}(y)$ and c is not a pole of f. Since P has no repeated roots, then B^2P has a pole at c of order $\geq 2m - 1$. Hence c is a pole of $\dfrac{\partial (B^2P)}{\partial y}$ of order $\geq 2m$ which contradicts with (3.3). $\square$

Corollary 3.3.11. *The elliptic function y such that*

$$y'^2 = y^3 + ay + b, \text{ where } a, b \in \mathbb{C}, a^3/27 + b^2/4 \neq 0 \tag{3.9}$$

is not liouvillian.

Remark 3.3.12. Hyperelliptic curves are generalisation of elliptic curves, and they were suggested by Koblitz [22] (1989) that they might be useful for public key cryptography. We note that there is not a group law (which exists on the points of an elliptic curve, see [23] or [26, Chapter 10]) on the points of a hyperelliptic curve; instead the divisor class group of the curve has been used, for details, see [17, Chapter 10].

To finish this chapter, we recall Rosenlicht's method in [46] for the proof of Corollary 3.3.11. We start with the following proposition.

Proposition 3.3.13. *([46, Proposition]) Let $\mathbf{k}$ be a differential field of characteristic zero, let n be a positive integer, and let f be a polynomial in several variables with coefficients in $\mathbf{k}$ and of total degree less than n. Then if the differential equation*

$$y^n = f(y, y', y'', \cdots)$$

has a solution in some liouvillian extension field of $\mathbf{k}$, it has a solution in an algebraic extension field E of $\mathbf{k}$.

Proof. It clearly suffices to prove this under the simpler assumption that the equation has a solution y in a differential extension field E of $\mathbf{k}$ which is a finite algebraic extension of a field $\mathbf{k}(t)$, where t is transcendental over $\mathbf{k}$ and either t' or t'/t is in $\mathbf{k}$. In this case, E is a field of algebraic functions of one variable over $\mathbf{k}$, see Section 1.3. Let P be a pole of t, that is, a $\mathbf{k}$-place of E such that $\text{ord}_P(t) < 0$. Then in either case $t' \in \mathbf{k}$ or $t'/t \in \mathbf{k}$, we get

$$\text{ord}_P(t'/t) \geq 0.$$

By [45, Lemma 1], the derivation on E is continuous in the topology of P. Hence case (1) of [46, Theorem] holds, and for any $x \in K$ we have

$$\text{ord}_P x' \geq \min(0, \text{ord}_P x).$$

Thus

$$\text{ord}_P y^{(m)} \geq \min(0, \text{ord}_P y) \text{ for all } m \geq 0.$$

If $\text{ord}_P y < 0$ then

$$\text{ord}_P f(y, y', y'', \cdots) \geq (n-1)\text{ord}_P y > n \cdot \text{ord}_P y = \text{ord}_P(y^n).$$

This is a contradiction. Therefore, $\text{ord}_P y^{(m)} \geq 0$ for all $m \geq 0$. Hence each $y^{(m)}(P)$ is finite (see Definition 1.3.7), therefore, algebraic over $\mathbf{k}$, with $y^{(m+1)}(P) = (y^{(m)}(P))'$. Thus $y(P)$ is a solution of the differential equation that is algebraic over $\mathbf{k}$. □

Proposition 3.3.14. *[46, page 372-373] The AODE (3.9) has no liouvillian solution over $\mathbb{C}$.*

Proof. It suffices to show that any element y of a liouvillian extension of the field $\mathbb{C}(x)$ of rational functions of the complex variable x which satisfies the AODE (3.9) is necessarily constant. Note that all elements of $\mathbb{C}$ are constant and that $C(x)$ is itself a liouvillian extension of $\mathbb{C}$, since $x' = 1$. As above, it remains only to show that if $\mathbf{k} \subset E$ are differential extension fields of $\mathbb{C}$, with E a finite algebraic extension of $\mathbf{k}(t)$, where t is transcendental over $\mathbf{k}$ and either $t' \in E$ or $t'/t \in E$, and if there exists a non-constant solution y of the AODE (3.9) in E, then there exists a non-constant solution that is algebraic over $\mathbf{k}$. Let the $\mathbf{k}$-place P of E be any pole of t, then by Theorem 3.3.13, $y(P)$ is a solution of the AODE (3.9) and we are all done unless it happens that $y'(P) = 0$. If this is the case, since we have $y' \neq 0$, the place P induces a nontrivial $\mathbb{C}$-place of the elliptic function field over $\mathbb{C}$ given by $\mathbb{C}(y, y')$, which is the associated algebraic function field over $\mathbb{C}$ of the cubic curve

$$w^2 = y^3 + ay + b. \tag{3.10}$$

The points of this curve (3.10) that are rational over $\mathbb{C}$ have a commutative group structure, see [26, Chapter 10], for instance. Each point of the curve that is rational over $\mathbb{C}$ produces a translation of the points of the curve which is an automorphism of the curve, equivalent to one of its function field $\mathbb{C}(y, y')$. By Galois theory of Kolchin [23, Chapter III, 6.], the latter one is a differential automorphism. For only a finite number of such differential automorphisms σ of $\mathbb{C}$ does the place induced by $\mathbb{C}$ on the cubic curve (3.10) go into one of the finite number of zeros of y'. Hence, for any σ distinct from a finite set we have $(\sigma y')(P) \neq 0$. That is $(\sigma y)'(P) \neq 0$. This follows $(\sigma y)(P)$ is a non-constant solution of the differential equation that is algebraic over $\mathbf{k}$ and the proof is complete. $\square$

Remark 3.3.15. The key point of the proof is to show the existence of the non-constant solution $y(P)$ such that $y'(P) \neq 0$. By induction, this is applied to the case $\mathbf{k} = \mathbb{C}$. Since $y(P)$ is algebraic over $\mathbb{C}$ then $y'(P) = 0$ which is a contradiction. Actually, the Rosenlicht's method has not been used in the next parts of this dissertation. However, we recall the method due to two reasons which provide the learners more materials to reference. First, we use the language of Section 1.3 for the case of solving AODEs. Last, we briefly introduce *weierstrassian elements* which satisfy the equation (3.9). It is clearly that a weierstrassian element is not liouvillian. More interesting properties of such weierstrassian elements can be found in [23, Chapter III, 6].

Conclusion

We have shown that a liouvillian solution of a first-order autonomous AODE of genus zero is necessary a rational liouvillian solution over $\mathbb{C}$ (see Lemma 3.2.2). In affirmative case, such a liouvillian solution is classified as an algebraic solution or a transcendental solution. Finally, these results lead to Algorithm `LiouSolAut` for determining liouvillian solutions of first-order autonomous AODEs of genus zero. This algorithm is complete since it terminates and if there is a liouvillian solution then its (implicit) form can be returned. In next chapter, we are going to consider if this idea can be applied for solving first-order non-autonomous AODEs.

Chapter 4

Liouvillian solutions of first-order AODEs

The content of this chapter is mainly based on the author's works in [37,38]. For the purpose considered in Section 4.4, let us denote $\mathbb{C}(z)$ be a differential field with the derivation $'$ is $\frac{d}{dz}$, i.e. the derivative respect to z, where $\mathbb{C}$ is the field of constants. A first-order AODE over $\mathbb{C}(z)$ is a differential equation of the form

$$F(Y, Y') = 0, \tag{4.1}$$

where F is an irreducible polynomial of $\mathbb{C}(z)[y, w]$.

Structure of the chapter is as follows. In Section 4.1, we take the method of [7,57] to dealt with the task of finding liouvillian solutions of the AODEs (4.1) whose genus are of zero. Using the theory of fields of algebraic functions of one variable, we prove that for determining liouvillian general solutions of the AODEs (4.1) of genus zero, working with the class of quasi-linear first-order ordinary differential equations (ODEs) is essentially enough. It turns out to be an algorithm for finding a liouvillian general solution of such a class of first-order AODEs. In Section 4.2, we give an algorithm to determine the reduced forms of certain first-order AODEs by means of power transformations. This leads to a method to solve first-order AODEs of positive genera in case of their reduced forms are of genus zero. Section 4.3 considers Möbius transformations. In Section 4.4, by means of change of variables, we present an approach for finding liouvillian solutions of a class of first-order AODEs with coefficients in a liouvillian extension of $\mathbb{C}(x)$.

4.1 Liouvillian solutions of first-order AODEs of genus zero

4.1.1 Associated differential equations

In this section, we take the method studied in [7,57] for finding liouvillian solutions of a first-order AODE (4.1) when its corresponding algebraic curve $F(y, w) = 0$ is rational. In this case, the curve has a proper parametrization

$$\mathcal{P}(t) = (u(t), v(t)) \in \overline{\mathbb{C}(z)}^2(t).$$

Such components of $\mathcal{P}(t)$

$$u = u(t), v = v(t),$$

can be seen as two algebraic functions of two variables z and t. To determine a solution of the AODE (4.1) via the proper parametrization $\mathcal{P}(t) = (u(t), v(t))$, we find a function $t = t(z)$ such that

$$\frac{d}{dz}u(t(z)) = v(t(z)).$$

Therefore

$$\frac{\partial u}{\partial t}t'(z) + \frac{\partial u}{\partial z} = v(t(z)). \tag{4.2}$$

Equation (4.2) motivates the following first-order ODE.

$$t'(z) = \frac{v(t) - \dfrac{\partial u(t)}{\partial z}}{\dfrac{\partial u(t)}{\partial t}}. \tag{4.3}$$

Definition 4.1.1. The ODE (4.3) is called the *associated differential equation* of the AODE (4.1) respect to a proper parametrization $\mathcal{P}(t) = (u(t), v(t))$.

Although the ODE (4.3) depends on the form of a proper parametrization of the corresponding rational curve, there is a relation between two associated differential equations which respect to different proper parametrizations.

Lemma 4.1.2. *[37, Lemma 3.3] Let $P(t)$ and $\tilde{\mathcal{P}}(t)$ be two proper parametrizations of F, then there is a change of variables $s = \dfrac{\alpha t + \beta}{\gamma t + \delta}$, where $\alpha, \beta, \gamma, \delta \in \overline{\mathbb{C}(z)}$, $\alpha\delta - \beta\gamma \neq 0$, between the two associated equations of F respect to $\mathcal{P}(t)$ and $\tilde{\mathcal{P}}(t)$.*

Proof. Let $P(t)$ and $\tilde{\mathcal{P}}(t)$ be two proper parametrizations of F. From Lemma 1.5.8 there is a linear function $\varphi(t) = \dfrac{\alpha t + \beta}{\gamma t + \delta}$, where $\alpha, \beta, \gamma, \delta \in \overline{\mathbb{C}(z)}$, $\alpha\delta - \beta\gamma \neq 0$, such that $\tilde{\mathcal{P}}(t) = \mathcal{P}(\varphi(t))$. The associated differential equation of F respect to $\tilde{\mathcal{P}(t)}$ is

$$t'(z) = \frac{v(\varphi(t)) - \dfrac{\partial u(\varphi(t))}{\partial z}}{\dfrac{\partial u(\varphi(t))}{\partial t}} = \frac{v(\varphi(t)) - \dfrac{\partial u(\varphi(t))}{\partial z}}{\dfrac{\partial u(\varphi(t))}{\partial \varphi}\dfrac{\partial \varphi}{\partial t}}. \tag{4.4}$$

From that, we obtain

$$(\varphi(t))'(z) = \frac{v(\varphi(t)) - \dfrac{\partial u(\varphi(t))}{\partial z}}{\dfrac{\partial u(\varphi(t))}{\partial \varphi}}. \tag{4.5}$$

By setting $s = \varphi(t)$, then $\mathcal{P}(s) = (u(s), v(s))$ is also a proper parametrization of F. Replacing $s = \varphi(t)$ in equation (4.5), we have the associated differential equation of F respect to $\mathcal{P}(s)$ is

$$s'(z) = \frac{v(s) - \dfrac{\partial u(s)}{\partial z}}{\dfrac{\partial u(s)}{\partial s}}. \tag{4.6}$$

Hence, there is a change $s = \varphi(t)$ between the two ODEs (4.4) and (4.6). □

4.1.2 Main results and an algorithm

The following theorem is the meat of this section.

Theorem 4.1.3. *[37, Theorem 3.4] A first-order AODE (4.1) of genus zero has a liouvillian general solution if and only if so does its associated ODE (4.3) respect to a certain proper parametrization $\mathcal{P}(s)$.*

Proof. It is obvious that if $s(z)$ is a liouvillian solution of the first-order ODE (4.3) over $\mathbb{C}(z)$, then $Y(z) = u(s(z))$ is a liouvillian solution of the first-order AODE (4.1) over $\mathbb{C}(z)$. In the reverse, we consider the two cases respect to the kind of liouvillian solutions.

Assume that $Y(z)$ is a liouvillian solution of the first-order AODE (4.1) which is transcendental over $\mathbb{C}(z)$. By Lemma 1.5.3, since $F(y, w) = 0$ is a rational curve, then the associated algebraic function field $\overline{\mathbb{C}(z)}(Y(z), Y'(z))$ is of genus zero, moreover, it is of the form $\overline{\mathbb{C}(z)}(t)$. Hence, there are functions $m(t)$ and $n(t)$ in $\overline{\mathbb{C}(z)}(t)$ such that

$$Y = m(t), \; Y' = n(t).$$

In this case, $\tilde{\mathcal{P}}(t) = (m(t), n(t))$ is a rational parametrization of F. Moreover

$$\frac{d}{dz}(m(t)) = n(t).$$

Hence, $t(z)$ is a solution of the associated ODE (4.3) respect to $\tilde{\mathcal{P}}(t)$. By Lemma 1.5.8, there is a rational function $\varphi(t) \in \overline{\mathbb{C}(z)}(t)$ such that

$$\mathcal{P}(\varphi(t)) = \tilde{\mathcal{P}}(t).$$

It follows $s = \varphi(t)$ is a solution of the ODE (4.3) respect to $P(s)$ (see Lemma 4.1.2). Since $t(z)$ is an algebraic function of $Y(z)$, and s is an algebraic function of $t(z)$, then $s(z)$ is a liouvillian solution of the ODE (4.3).

Assume that $Y(z)$ is an algebraic general solution of the AODE (4.1) over $\mathbb{C}(z)$. By [57, Lemma 5.2], there is an algebraic solution $s(z)$ of the ODE (4.3) such that $Y(z) = u(s(z))$. The proof is complete. □

Remark 4.1.4. In Theorem 4.1.3, the property of having a liouvillian solution of an AODE (4.1) is independent of the choice of proper parametrizations of its corresponding algebraic curve. From [57, Theorem 4.3], there is an optimal parametrization $\mathcal{P}(t)$ such that the ODE (4.3) is of the form

$$t' = \frac{dt}{dz} = f(z,t) \in \mathbb{C}(z,t). \tag{4.7}$$

Hence, without loss of generality, when considering an associated differential equation of the AODE (4.1) of genus zero, we may consider it in the form (4.7).

From the above results, we give a pseudo-code algorithm for finding a liouvillian general solution of an AODE (4.1).

Algorithm `LiouSol`

Input: A first-order AODE $F(Y, Y') = 0$ (4.1) of genus zero.

Output: A liouvillian general solution of $F(Y, Y') = 0$ if any.

---

1. Find an optimal parametrization $\mathcal{P}(t) = (u(t), v(t)) \in (\mathbb{C}(z,t))^2$ of $F(y, w) = 0$.
2. Compute the associated ODE (4.7) respect to $\mathcal{P}(t)$.
3. If the ODE (4.7) has a liouvillian general solution $t(z)$, then **return** "$Y(z) = u(t(z))$ is a liouvillian general solution of $F(Y, Y') = 0$".
4. Else, **return** "$F(Y, Y') = 0$ has no liouvillian general solution".

4.1.3 An investigation of first-order ODEs (4.7) and examples

In Kamke's book, more than 70 percent of the ODEs listed in [21, page 355-391] are first-order AODEs of genus zero, more precisely, a classification of such AODEs can be found in [6, Appendix]. In this situation, Algorithm `LiouSol` can be applied for finding their liouvillian solutions. Solving the ODE (4.7) is an intrinsic step of Algorithm `LiouSol`, unfortunately, up to this time we do not know if an ODE (4.7) can be solved completely. In below, we consider some cases.

Proposition 4.1.5. *[37, Proposition 3.6] If the ODE (4.7) is of the form*

$$t' = b(z)t + c(z), \tag{4.8}$$

where $b(z), c(z) \in \mathbb{C}(z)$ then it always has a liouvillian general solution.

Proof. In fact, such ODE (4.8) follows the differential equation

$$\left(t \exp \int -b(z)\right)' = c(z) \exp \int -b(z)$$

whose solution is

$$t(z) = \left(c + \int \left[c(z) \exp \int -b(z)\right]\right) \exp \int b(z),$$

(where constants $c \in \mathbb{C}$) which belongs to the following liouvillian extension

$$\mathbb{C}(z) \subset \mathbb{C}\left(z, \exp \int -b(z)\right) \subset \mathbb{C}\left(z, \exp \int -b(z), \int \left[c(z) \exp \int -b(z)\right]\right). \qquad \square$$

Remark 4.1.6. The ODE (4.8) may not have an elementary solution, for instance, see the case with $b(z) = 2z$ and $c(z) = 1$ in Example 1.1.19. The more general case of the ODE (4.8), i.e. the Risch differential equation problem to decide whether the ODE

$$t' = ft + g, \text{ where } f, g \in E$$

has a solution in E and find one if any, can be found in [4, Chapter VI] or in [41].

Proposition 4.1.7. *[37, Proposition 3.8] Assume that the ODE (4.7) is of the form a Riccati equation*

$$t' = a(z)t^2 + b(z)t + c(z), \tag{4.9}$$

where $a(z), b(z), c(z) \in \mathbb{C}(z)$ and $a(z) \neq 0$. Then we can determine if it has a liouvillian solution or not.

Proof. By setting

$$t = -\frac{u'}{a(z)u}, \tag{4.10}$$

the ODE (4.9) is transformed into the second order homogeneous ODE

$$u'' - \frac{a(z)b(z) + a'(z)}{a(z)}u' + c(z)a(z)u = 0. \tag{4.11}$$

The problem of determining liouvillian solutions of the ODE (4.11) has been solved completely by the works of Kovacic in [25]. Moreover, the formula (4.10) ensures that the ODE (4.9) has a liouvillian solution if and only if so does the ODE (4.11). In fact, if $t(z)$ is a liouvillian solution of the ODE (4.9) then $u(z)$ is a liouvillian solution of the ODE (4.11) since

$$\frac{u'}{u} = -a(z)t$$

belongs to a liouvillian extension of $\mathbb{C}(z)$. The inverse statement is clearly by using similar argument. In a special case, if the ODE (4.9) is autonomous, by Lemma 3.3.7, it always has a liouvillian general solution. □

Proposition 4.1.8. *[37, Proposition 3.9] If the ODE (4.7) is autonomous, then we can determine if it has a liouvillian solution or not.*

Proof. It is clearly that our statement is due to [56, Proposition 3.1] and [41, Main Theorem]. From [36], the liouvillian solution of the autonomous ODE (4.7) is either an algebraic function over $\mathbb{C}(z)$ or over $\mathbb{C}(\exp az)$. Therefore, a liouvillian general solution of the AODE (4.1) is either an algebraic function over $\mathbb{C}(z)$ or an algebraic function over $\mathbb{C}(z)(\exp az)$. □

Concerning to other forms of solutions, there are following results.

Proposition 4.1.9. *[37, Proposition 3.10] If the ODE (4.7) admits a transcendental meromorphic solution or a rational general solution, then it is necessarily of the form a Riccati equation (4.9).*

Proof. First, the condition for the ODE (4.7) having a transcendental meromorphic solution is well-known in analysis, and its proof can be found in [10, 29]. If $t(z)$ is such a solution, then $u(t(z))$ is a transcendental meromorphic solution of the AODE (4.1). Next, [57, Section 5] has shown that the rational general solutions of the ODE (4.7) can not be found outside a Riccati equation. Note that, the problem of finding a rational general solution of a parametrizable AODE (4.1) is solved completely by the works of [2, 7, 25, 57]. If $t(z)$ is a rational general solution of the ODE (4.7), then $u(t(z))$ is a rational general solution of the AODE (4.1). □

Remark 4.1.10. Besides above cases, there are other situations that we can rely.

- If the ODE (4.7) drops in the form of the ones considered in [21, page 293-354] then the method for solving them in [21, page 1-32] can be applied. In this case, we may build liouvillian extensions (see [4]) to decide if our solutions can be the liouvillian ones.

- One more case, the method of finding liouvillian first integrals (see [55, page 673]) can be applied for determining liouvillian solution. In fact, the ODE (4.7) can be written by following formula

$$\frac{dt}{dz} = \frac{M(z,t)}{N(z,t)} \tag{4.12}$$

  where $M(z,t), N(z,t) \in \mathbb{C}[z,t]$, which induces the differential form $(Mdz - Ndt)$. By the works of [9, 40, 55], we can determine if such the differential form has a liouvillian first integral $I(z,t)$ which generates the general solution $I(z,t) = c$ of the ODE (4.12), where $I(z,t)$ is a liouvillian function (see [52, 55]).

- Using software computation, MAPLE may help us process our method as follows.

```
>  with(algcurves):
>  F:=F(y,w);
>  g:=genus(F,y,w);
>  P:=parametrization(F,y,w,t); # (only exists when g = 0)
>  f:=simplify(P[2]-diff(P[1],z)/diff(P[1],t));  # (assoc.  ODE)
```

  Assume that $\mathrm{f} := \mathrm{f}(z,t)$ then we solve it by following commands.

```
>  infolevel[dsolve] := 2; # (info.  of the efficient method)
>  dsolve(t'- f(z,t)=0); # (sol.  the associated ODE)
```

We conclude this section by some examples which illustrate Algorithm `LiouSol` and the above considerations. In an affirmative case, by using MAPLE 2022 with above commands, the ODE

$$t' = t^2 - 2zt$$

has a liouvillian general solution (the Bernoulli method is used, see [21, page 19])

$$t(z) = \frac{-2\exp(-z^2)}{\sqrt{\pi}\,\mathrm{erf}(z) - 2c}$$

since it belongs to a liouvillian extension E of $\mathbb{C}$ as follows

$$E = \mathbb{C}\left(z, \exp(-z^2), \int \exp(-z^2)dz\right) \supset \mathbb{C}\left(z, \exp(-z^2)\right) \supset \mathbb{C}(z) \supset \mathbb{C}.$$

In the other case, by [9], the following ODE (see [9, Example 7])

$$t' = -\frac{3 + t^2 z^4}{z^4}$$

has a liouvillian general solution $I(z,t) = c$, where $I(z,t)$ is a liouvillian first integral

$$I(z,t) = \frac{1}{\sqrt{3}} \arctan\left(\frac{z^2 t - z}{\sqrt{3}}\right) - \frac{1}{z}.$$

In some cases, the method of [9] may help us find the solution which can not be detected by MAPLE 2022, for instance, see [9, Example 2.1]. However, we note that a general solution of a certain ODE (4.7) induced by such a liouvillian first integral is not necessary a liouvillan solution. In fact, a general solution $I(z,t) = c$ of the ODE

$$t' = \frac{t^2}{z(t-z)}$$

induced by the liouvillian first integral

$$I(z,t) = \log t - \frac{t}{z},$$

is not liouvillian, see [45, page 22].

Example 4.1.11. Consider the first-order AODE

$$F(Y, Y') = -z^3 Y^3 + z^2 Y'^2 - 2z^2 Y^2 + 2zYY' - zY + Y^2 = 0. \tag{4.13}$$

Non-constant solutions of the AODE (4.13) determined by MAPLE 2022 are not explicit (it involves integral signs and Root-Of). On the other hand, by MAPLE 2022, the corresponding curve of (4.13) has a proper parametrization

$$\mathcal{P}(t) = \left(\frac{z^2 t^2 + 2zt + 1}{z(t^2 + 2zt + z^2)}, \frac{t^3 z^4 + 4t^2 z^3 + tz^4 - t^2 z + 3tz^2 + z^3 - t}{z^2(t^3 + 3t^2 z + 3zt^2 + z^3)}\right).$$

The associated ODE of (4.13) respect to $\mathcal{P}(t)$ is a Riccati equation (4.9)

$$t'(z) = \frac{(z^2 - 1)t^2 + 4tz + (z^2 + 3)}{2(z^2 - 1)}. \tag{4.14}$$

By the change (see Lemma 4.1.2)

$$s = \frac{zt + 1}{t + z},$$

the ODE (4.14) is transformed into an autonomous one

$$s'(z) = \frac{s^2 + 1}{2}, \tag{4.15}$$

which corresponds with the proper parametrization

$$\mathcal{P}(s) = \left(\frac{s^2}{z}, \frac{(s^2z - s + z)s}{z^2}\right).$$

A liouvillian general solution of (4.15) is

$$s(z) = i\frac{1 - \exp i(z+c)}{1 + \exp i(z+c)}, i^2 = -1.$$

Hence, a liouvillian general solution of the original AODE (4.13) is

$$Y(z) = -\frac{(1 - \exp i(z+c))^2}{z(1 + \exp i(z+c))^2}.$$

Example 4.1.12. [21, 464, page 374] Consider first-order AODE

$$YY'^2 + 2zY' - Y = 0. \tag{4.16}$$

The corresponding curve of (4.16) has a proper parametrization

$$\mathcal{P}(t) = \left(-\frac{2zt}{t^2 - 1}, t\right).$$

The associated ODE of the AODE (4.16) respect to $\mathcal{P}(t)$ is an Abel equation of the first kind (for details, see [21, page 24])

$$t' = \frac{t^3 - t}{2z}, \tag{4.17}$$

which has an algebraic general solution

$$t(z) = \frac{1}{\sqrt{cz+1}}.$$

Hence, an algebraic general solution of (4.16) is

$$Y(z) = \frac{2\sqrt{cz+1}}{c}.$$

Remark 4.1.13. It is clear that Algorithm `LiouSol` can not be applied directly to first-order AODEs of positive genera since they have no rational parametrization. To deal with the AODEs of positive genera, radical parametrizations of algebraic curves introduced in [47] can be used, see [18, PROCEDURE 1]. This method leads to the problem of solving a radical integral. We note that [48, Algorithm 4.12] can be applied for finding rational parametrizations from the radical ones, by these, a radical integrand is converted into the one which is rational, see [48, Example 4.18]. Unfortunately, this algorithm is only valid for rational algebraic curves, hence, it is not applicable in these non-rational cases. In Section 4.2, we give a novel approach to determine liouvillian solutions of first-order AODEs of positive genera whose final steps relies on rational parametrizations.

4.2 Power transformations and their applications

We define power transformations among polynomials. Based on this, we give an algorithm to determine the reduced forms of first-order AODEs (4.1). In application, we present a method for solving first-order AODEs (4.1) of positive genera in the case of the reduced forms are of genus zero.

4.2.1 Power transformations

Let $G(u, u')$ be a polynomial in $\mathbb{C}(z)[u, u']$, and let us write it in the form

$$G(u, u') = G_0(u, u') + G_1(u, u') + \ldots + G_d(u, u'), \tag{4.18}$$

where

$$G_k(u, u') = \sum_{i+j=k} c_{ij} u^i u'^j, c_{ij} \in \mathbb{C}(z)$$

is the homogeneous component of degree k of G.

Definition 4.2.1. [37, Definition 4.1] A *power transformation* is a transformation of the form

$$u = Y^n, u' = nY^{n-1}Y', 2 \leq n \in \mathbb{N}. \tag{4.19}$$

Substituting (4.19) into $G_k(u, u')$, we obtain

$$\sum_{i+j=k} c_{ij} Y^{ni} (nY^{n-1}Y')^j = \sum_{i+j=k} c_{ij} n^j Y^{(n-1)k} Y^i Y'^j, \tag{4.20}$$

which is a homogeneous polynomial of degree nk. Hence, (4.18) is transformed into

$$G(u, u') = G\left(Y^n, nY^{n-1}Y'\right) = \sum_{k=0}^{d} \sum_{i+j=k} c_{ij} n^j Y^{(n-1)k} Y^i Y'^j. \tag{4.21}$$

Let k_0 be the lowest degree of the non-zero homogeneous component of G, then by factorization we obtain

$$G(u, u') = Y^{(n-1)k_0} \sum_{k=k_0}^{d} \sum_{i+j=k} c_{ij} n^j Y^{(n-1)(k-k_0)} Y^i Y'^j = Y^{(n-1)k_0} F(Y, Y'), \tag{4.22}$$

where

$$F(Y, Y') = \sum_{k=k_0}^{d} \sum_{i+j=k} c_{ij} n^j Y^{(n-1)(k-k_0)} Y^i Y'^j = F_{n_{k_0}}(Y, Y') + \cdots + F_{n_d}(Y, Y').$$

Above considerations lead to the following lemma.

Lemma 4.2.2. *[37, Lemma 4.2] Let $G(u,u')$ and $F(Y,Y')$ are two polynomials over $\mathbb{C}(z)$. If there is a power transformation (4.19) such that the formula (4.22) is satisfied, then the followings hold.*

1. *For each $k \geq k_0$, the polynomial*

$$F_{n_k}(Y,Y') = \sum_{i+j=k} c_{ij} n^j Y^{(n-1)(k-k_0)} Y^i Y'^j$$

is homogeneous of degree $n_k = n(k-k_0)+k_0$, and n_{k_0} is the lowest degree among the non-zero homogeneous components of F. Moreover, $n_{k_0} = k_0$.

2. *Let n_{k_1} and n_{k_2} be the degrees of two different homogeneous components of F. Then n is a common divisor of $(n_{k_1} - n_{k_0})$ and $(n_{k_2} - n_{k_0})$.*

3. *If F is an irreducible polynomial then so is G. In this case, if F has genus zero then the genus of G is zero too. Moreover, the reverse of these two properties are not true.*

Proof. 1. From (4.22), let $k \geq k_0$ and $n \geq 2$, each of above polynomials has degree

$$n_k = (n-1)(k-k_0) + k = n(k-k_0) + k_0.$$

Clearly that, $n_{k_0} = k_0$. If $k > k_0$ then $n_k > n_{k_0}$. By abuse of notation, we also call k_0 the lowest degree of the non-zero homogeneous component of F.

2. From above expression, we can write

$$n_{k_1} - n_{k_0} = n(k_1 - k_0); \quad n_{k_2} - n_{k_0} = n(k_2 - k_0).$$

Clear that, n is a common divisor of $(n_{k_1} - n_{k_0})$ and $(n_{k_2} - n_{k_0})$. To avoid trivial cases, we always assume that F is a non-homogeneous polynomial (so is G).

3. First, assume that F is irreducible, then G is irreducible too. In fact, if G is reducible, it can be written (see (4.22))

$$G(u,u') = \tilde{G}_1(u,u')\tilde{G}_2(u,u') = Y^{(n-1)k_{01}} \tilde{F}_1(Y,Y') Y^{(n-1)k_{02}} \tilde{F}_2(Y,Y'),$$

where $k_{01} + k_{02} = k_0$. By comparing with (4.22), we obtain following contradiction

$$F(Y,Y') = \tilde{F}_1(Y,Y')\tilde{F}_2(Y,Y').$$

Hence, G is irreducible. In addition, if $F = 0$ is of genus zero, there is a rational pair $(r(t), s(t)) \in (\mathbb{C}(z)(t))^2$ that $F(r(t), s(t)) = 0$. By (4.22), $G(r^n(t), nr^{n-1}(t)s(t)) = 0$.

That means $(r^n(t), nr^{n-1}(t)s(t)) \in (\mathbb{C}(z)(t))^2$ is a rational parametrization of $G = 0$. Hence, $G = 0$ is of genus zero. The reverse is not true. For instance, we consider the first-order AODE in Example 4.1.11

$$G(u, u') = -zu + (u^2 - 2z^2u^2 + 2zuu' + z^2u'^2) - z^3u^3.$$

In this case, $k_0 = 1$. Choose $n = 2$, by substituting

$$u = Y^2, u' = 2YY',$$

and comparing with (4.22) we obtain

$$G(u, u') = G(Y^2, 2YY') = YF(Y, Y'),$$

where

$$F(Y, Y') = Y\left(-z + (Y^2 - 2z^2Y^2 + 4zYY' + 4z^2Y'^2) - z^3Y^4\right)$$

is a reducible polynomial. This follows the polynomial $F(Y, Y')$ obtained by putting a transformation (4.19) into an irreducible polynomial $G(u, u')$ in the formula (4.22) may not be irreducible. Moreover, even if F is irreducible in the case of $G = 0$ has genus zero, the algebraic curve $F = 0$ is not necessarily rational, for instance, see Example 4.2.14 and also Example 4.2.16. □

4.2.2 Reduced forms by power transformations

We aim to investigate all of the irreducible polynomials $G(u, u')$ whose power transformations arrive at an irreducible polynomial $F(Y, Y')$. We start with following definition.

Definition 4.2.3. [37, Definition 4.3] Let $F(Y, Y')$ be an irreducible polynomial. Let HD_F be the set of degrees of the non-zero homogeneous components of F, and let $k_0 = n_{k_0}$, see (1.) in Lemma 4.2.2, be the smallest element of HD_F. We define the set

$$\mathbb{D}_F = \{n \geq 2 \mid \ n \text{ is a common divisor of all } (m - k_0) \text{ for } m \in \text{HD}_F\}. \qquad (4.23)$$

Suppose that $\mathbb{D}_F \neq \emptyset$, and let $n \in \mathbb{D}_F$. We say such n *induces* a transformation of the form (4.19) if there is an irreducible polynomial $G(u, u')$ such that the formula (4.22) is satisfied. In this case, we say F is *transformed from* G by the transformation (4.19) respect to n. We define the set

$$\mathbb{P}_F = \{n \in \mathbb{D}_F \mid n \text{ induces a transformation } (4.19)\}. \qquad (4.24)$$

Clearly that, $\mathbb{P}_F \subseteq \mathbb{D}_F$. If $\mathbb{D}_F = \emptyset$, then $\mathbb{P}_F$ is an empty set too.

Lemma 4.2.4. *[37, Lemma 4.4] Let $F(Y,Y')$ be an irreducible homogeneous polynomial then $\mathbb{D}_F$ is an infinite set. Moreover, $\mathbb{D}_F$ coincides with $\mathbb{P}_F$.*

Proof. Since F is homogeneous, we can write it as follows

$$F(Y,Y') = \sum_{i+j=k_0} c_{ij} Y^i Y'^j.$$

From Definition 4.2.3, $\mathrm{HD}_F = \{k_0\}$. This follows $\mathbb{D}_F = \mathbb{N}\backslash\{0,1\}$ is an infinite set. For any $n \in \mathbb{D}_F$, we have

$$Y^{(n-1)k_0} F(Y,Y') = Y^{(n-1)(i+j)} \sum_{i+j=k_0} c_{ij} Y^i Y'^j = \sum_{i+j=k_0} \frac{c_{ij}}{n^j} Y^{ni} (nY^{n-1}Y')^j.$$

By setting

$$a_{ij} = \frac{c_{ij}}{n^j} \text{ and } G(u,u') = \sum_{i+j=k_0} a_{ij} u^i u'^j,$$

then $F(Y,Y')$ is transformed from $G(u,u')$ by a power transformation (4.19) which is induced by n. This follows that $n \in \mathbb{P}_F$, and the set $\mathbb{D}_F$ coincides with $\mathbb{P}_F$. Moreover, although there are many power transformations (4.19), they do not change the degree of G versus F. □

Lemma 4.2.5. *[37, Lemma 4.5] Let $F(Y,Y')$ be an irreducible non-homogeneous polynomial. Then $\mathbb{D}_F$ is either a finite set or an empty one. Moreover, it is different from $\mathbb{P}_F$.*

Proof. Since F is a non-homogeneous polynomial then $|\mathrm{HD}_F| \geq 2$. By the observations of (4.23) and part (2.) in Lemma 4.2.2, then $\mathbb{D}_F$ is either a finite set or an empty one. Moreover, we show $\mathbb{P}_F$ is different from $\mathbb{D}_F$ by considering an irreducible polynomial

$$F(Y,Y') = Y^4 + Y'^4 - 1.$$

In this case, $\mathrm{HD}_F = \{0,4\}$. This follows $k_0 = 0$ and $\mathbb{D}_F = \{2,4\}$. In finally, by computation we find that $\mathbb{P}_F = \emptyset$. □

Definition 4.2.6. [37, Definition 4.6] An irreducible non-homogeneous polynomial $F(Y,Y')$ is called a *reduced form* if $\mathbb{P}_F = \emptyset$. Otherwise, we call F a non-reduced form.

Remark 4.2.7. From Lemma 4.2.5, since F is a non-homogeneous polynomial then $\mathbb{D}_F$ is a finite set or an empty one. Since $\mathbb{P}_F \subseteq \mathbb{D}_F$ then it is a finite set or an empty one too. Hence, if F is a non-reduced form then $\mathbb{P}_F$ is finite.

Theorem 4.2.8. *[37, Theorem 4.8] Let F be a non-reduced form. Let n be the greatest element of $\mathbb{P}_F$ and G be an irreducible polynomial such that F is transformed from G respect to the power transformation induced by n. Then G is of a reduced form.*

Proof. Since F is non-reduced then $\mathbb{P}_F \neq \emptyset$ and it is a finite set. There exists a greatest element n of $\mathbb{P}_F$ and an irreducible polynomial G respect to the power transformation induced by n. Suppose that such G is not a reduced form, then $\mathbb{P}_G \neq \emptyset$, and there is $2 \leq n_1 \in \mathbb{P}_G$. In this case, from Definition 4.2.3 and formula (4.22), we have

$$H(v, v') = H(u^{n_1}, n_1 u^{n_1-1}) = u^{(n_1-1)k_0} G(u, u'),$$

and

$$G(u, u') = G(Y^n, nY^{n-1}) = Y^{(n-1)k_0} F(Y, Y').$$

That means

$$H(v, v') = Y^{(nn_1-1)k_0} F(Y, Y').$$

Hence, we obtain $n < nn_1 \in \mathbb{P}_F$ which is a contradiction. Hence, G is a reduced form. The proof is complete. □

Remark 4.2.9. From the above formulas in Theorem 4.2.8, we find that the greatest element n is unique and it is divisible for all elements of $\mathbb{P}_F$. Therefore, such polynomial G is also a unique one respect to F. In this case, it is called *the reduced form* of F.

From the above results, there is an algorithm to determine the reduced form G from a certain non-homogeneous polynomial F.

Algorithm `RedPol`

Input: An irreducible non-homogeneous polynomial $F(Y, Y')$.

Output: The reduced form of F and the transformation (4.19) if any.

---

1. Rewrite F in non-zero homogeneous components to determine HD_F.
2. Find k_0 and compute $\mathbb{D}_F$.
3. Determine $\mathbb{P}_F$ (see Definition 4.2.3).
4. If $\mathbb{P}_F = \emptyset$, then **return** "$F(Y, Y')$ is of reduced form and there is no power transformation (4.19)".
5. Else, let $n = \max \mathbb{P}_F$ (i.e. the greatest element of $\mathbb{P}_F$), then **return** "The reduced form $G(u, u')$ and the power transformation (4.19) respect to n".

Remark 4.2.10. Algorithm `RedPol` lets us know if there is a transformation (4.19) such that an irreducible non-homogeneous polynomial $F(Y, Y')$ is transformed from the reduced form $G(u, u')$. In the affirmative case, the lager n is, the smaller the degree of G is. In addition, let $F(Y, Y')$ be defining polynomial of the AODE (4.1), then such reduced form $G(u, u')$ can be seen as defining polynomial of the AODE

$$G(u, u') = 0. \tag{4.25}$$

In this case, we say that the AODE (4.25) is *reduced* and the AODE (4.1) is *transformed from* the reduced AODE (4.25) by a transformation (4.19) respect to n. In other words, we can say that the AODE (4.25) is the *reduced form* of the AODE (4.1).

To conclude this section, the following theorem ensures that a transformation (4.19) preserves the property of having a liouvillian solution between a certain first-order AODE (4.1) and its reduced form (4.25).

Theorem 4.2.11. *[37, Theorem 4.11] Suppose that an AODE (4.1) is transformed from a reduced AODE (4.25) by a power transformation (4.19) (respect to $n \geq 2$). Then the AODE (4.1) has a liouvillian solution if and only if so does the AODE (4.25). Moreover, if η is a liouvillian solution of the AODE (4.25), then there is a liouvillian solution ξ of the AODE (4.1) which satisfies following equation*

$$Y^n - \eta = 0. \tag{4.26}$$

Proof. From the hypothesis, there is $n \geq 2$ and $k_0 \geq 0$ such that

$$Y^{(n-1)k_0} F(Y, Y') = G(Y^n, nY^{n-1}Y') = G(u, u').$$

Let η be a liouvillian solution of $G(u, u') = 0$, then we consider two cases. First, suppose that $\eta = 0$, then we obtain

$$k_0 = \min \mathrm{HD}_G \geq 1.$$

From Lemma 4.2.2, we find that the AODE $F(Y, Y') = 0$ has a solution $\xi = 0$ since

$$\min \mathrm{HD}_F = \min \mathrm{HD}_G = k_0 \geq 1.$$

Next, suppose that η be a non-zero solution. Let ξ be a solution of the equation (4.26), then $\xi^{(n-1)k_0} \neq 0$ and ξ is a non-zero solution of $F(Y, Y') = 0$ by the above relation. Moreover, such ξ is a liouvillian solution since it is algebraic over $\mathbb{C}(\eta)$. From the two cases considered, we can say that if η is a liouvillian solution of the AODE (4.25) then there is a liouvillian solution ξ of the AODE (4.1) which satisfies the equation (4.26).

In the reverse, if ξ is a liouvillian solution of $F(Y, Y') = 0$ then it is not hard to see that $\eta = \xi^n$ is a liouvillian solution of $G(u, u') = 0$. The proof is complete. □

4.2.3 Applications

This section is devoted for illustrating the useful aspects when we are combining Algorithm `RedPol` and Algorithm `LiouSol`. In particular, for solving an AODE (4.1), we consider its reduced form, an AODE (4.25). Then we use the relation shown in Theorem 4.2.11 to return the solutions of the original AODE (4.1) if there is any.

First, we show that in some cases of solving an AODE (4.1) of genus zero, the reduced AODE (4.25) may be more suitable for applying Algorithm `LiouSol` since its degree is really decreased versus the one of the original AODE.

Example 4.2.12. [21, I·490, page 379] Consider the first-order AODE of genus zero

$$F(Y, Y') = Y^2Y'^2 - 2zYY' + 2Y^2 - z^2 + a = 0. \tag{4.27}$$

MAPLE 2022 finds a proper parametrization of the algebraic corresponding curve of the AODE (4.27) after hundreds of seconds, and its output is not suitable for using Algorithm `LiouSol` due to such parametrization occupies approximately eight lines and the associated ODE respect to it contains more than thirty lines in a Maple's page. In order to apply Algorithm `RedPol`, we rewrite the AODE (4.27) into homogeneous components

$$F(Y, Y') = (a - z^2) + (2Y^2 - 2zYY') + Y^2Y'^2 = 0.$$

In this case, $\mathrm{HD}_F = \{0, 2, 4\}$. This follows $k_0 = 0$ and $\mathbb{D}_F = \{2\}$. From that, we determine $\mathbb{P}_F = \{2\}$, and hence $n = \max \mathbb{P}_F = 2$. The transformation (4.19) respect to such $n = 2$ is

$$u = Y^2, u' = 2YY',$$

and the AODE (4.27) is transformed from the reduced form

$$G(u, u') = a - z^2 + (2u - zu') + \frac{u'^2}{4} = 0. \tag{4.28}$$

By MAPLE 2022, the corresponding curve of (4.28) has a proper parametrization

$$\left(\frac{-t^2 + 4zt + 4z^2 - 4a}{8}, t \right).$$

By Algorithm `LiouSol`, a liouvillian solution of (4.28) is

$$-(-2z + c)^2 + 4z(-2z + c) + 4z^2 - 4a - 8u = 0.$$

From Theorem 4.2.11, a liouvillian general solution of (4.27) is

$$8z^2 - 8cz + c^2 + 4a + 8Y^2 = 0.$$

Remark 4.2.13. First-order AODEs of genus zero in Kamke's collection which are similar to the above example are listed [21, I·431, 464, 465, 466, 467, 468, 469, 470, 474, 475, 476, 477, 481, 486, 488, 489, 490, 491, 497, 499, 500, 508, 510.]

The following example is based on [21, I·431] which shows that there exists a first-order AODE of positive genus whose reduced form is an AODE of genus zero.

Example 4.2.14. Consider the first-order AODE of genus two

$$F(Y, Y') = (Y^2 + 9z^2Y'^2) - Y^8 = 0. \tag{4.29}$$

From the AODE (4.29), $\mathrm{HD}_F = \{2, 8\}$. This follows $k_0 = 2$ and $\mathbb{D}_F = \{2, 3, 6\}$. By computation, we determine

$$\mathbb{P}_F = \{2, 3, 6\}.$$

Hence $n = \max \mathbb{P}_F = 6$. The transformation respect to $n = 6$ is

$$u = Y^6, u' = 6Y^5Y',$$

and the AODE (4.29) is transformed from the rational AODE

$$G(u, u') = (u^2 + \frac{1}{4}z^2u'^2) - u^3 = 0. \tag{4.30}$$

The corresponding algebraic curve of (4.30) has a proper parametrization

$$\left(\frac{t^2z^2 + 4}{4}, \frac{t(t^2z^2 + 4)}{4}\right).$$

By Algorithm `LiouSolAut`, a liouvillian general solution of (4.30) is

$$u = 1 + \tan^2(c - \log z).$$

From Theorem 4.2.11, a liouvillian general solution of the AODE (4.29) is

$$Y^6 - 1 - \tan^2(c - \log z) = 0.$$

Remark 4.2.15. First-order AODEs of positive genera in Kamke's collection whose reduced form are first-order AODEs of genus zero are listed [21, I·482, 485, 487, 504, 509, 541, 542, 543, 544].

We may avoid solving a radical integral when finding solutions of a first-order AODE of genus one in [18] by considering its reduced form.

Example 4.2.16. (Example 2.4.11) Consider a first-order AODE of genus one

$$F(Y, Y') = -Y^3 - 4Y^5 + 4Y^7 - 2Y' - 8Y^2Y' + 8Y^4Y' + 8YY'^2 = 0. \tag{4.31}$$

The defining polynomial of the AODE (4.31) can be written as

$$F(Y, Y') = -2Y' + (-Y^3 - 8Y^2Y' + 8YY'^2) + (-4Y^5 + 8Y^4Y') + 4Y^7.$$

In this case, $\mathrm{HD}_F = \{1, 3, 5, 7\}$. This follows $k_0 = 1$ and $\mathbb{D}_F = \{2\}$. Hence, we obtain that $\mathbb{P}_F = \{2\}$ and $n = \max \mathbb{P}_F = 2$. The transformation (4.19) respect to $n = 2$ is

$$u = Y^2, u' = 2YY',$$

and the AODE (4.31) is transformed from the AODE of genus zero

$$G(u, u') = -u' + (-u^2 - 4uu' + 2u'^2) + (-4u^3 + 4u^2u') + 4u^4 = 0. \tag{4.32}$$

The corresponding algebraic curve of (4.32) has a proper parametrization

$$\tilde{\mathcal{P}}(t) = \left(-\frac{2(17t + 1)t}{365t^2 + 38t + 1}, \frac{81209t^4 + 19380t^3 + 1726t^2 + 68t + 1}{2(133225t^4 + 27740t^3 + 2174t^2 + 76t + 1)} \right).$$

The associated ODE respect to $\tilde{\mathcal{P}}(t)$ is

$$t' = -\frac{(17t + 1)^2}{4}, \tag{4.33}$$

which has a liouvillian general solution

$$t = -\frac{17(z + c) - 4}{289(z + c)}.$$

Therefore, a liouvillian general solution of (4.32) is

$$u = \frac{289(z + c) - 68}{289(z + c)^2 - 714(z + c) + 730}.$$

From Theorem 4.2.11, a liouvillian general solution of (4.31) is

$$Y^2 - \frac{289(z + c) - 68}{289(z + c)^2 - 714(z + c) + 730} = 0.$$

In the last part, we aim to consult more of liouvillian solutions of the AODEs of any genera. If we put the transformations (4.19) (respect to n) into a rational AODE $G(u, u') = 0$, then we may obtain the increasing of the genus of $F(Y, Y') = 0$. This idea leads to a method of generating an AODE of any positive genus from a rational one. Moreover, their liouvillian solutions (if any) can be connected by

Theorem 4.2.11. Clearly, the property of having liouvillian solutions is based on the original AODE whose genus is of zero. In below with the helping of Maple, we present a procedure to illustrate the idea and apply it to above examples.

```
> with(algcurves):
> Testgenus := proc(G, u, v, n, k0)
> local F, y, w;
> F := simplify(subs(u = y^n, v = n*y^(n - 1)*w, G)/y^((n - 1)*k0));
> genus(F, y, w);
> end proc; # the procedure for determining genus of F
> G:= a - z^2 + 2*u - z*v + v^2/4;     # Example 4.2.12
> for i from 2 to 10 do
> Testgenus(G, u, v, i, 0);  #(k0= 0)
> od;
> 0  1 1 2 2 3 3 4 4  5 5 6 6 7 7 8 8 9 9      # result
> G:=u^2+ z^2*v^2/4-u^3;     # Example 4.2.14
> for i from 2 to 20 do
> Testgenus(G, u, v, i, 2);   # (k0= 2)
> od;
> 0  1 1 2 2 3 3 4 4  5 5 6 6 7 7 8 8 9 9      # result

> G:=2*v^2+4*u^2*v-4*u*v-v+4*u^4-4*u^3-u^2;  # Example 4.2.16
> for i from 1 to 10 do
> Testgenus(G, u, v, i, 1);  #(k0= 1)
> od;
> 0 1 2 3 4 5 6 7 8 9   # result
```

The above computation shows that there are first-order AODEs of positive genera which obtain a non-constant liouvillian solution. Unfortunately, we are not in control of the change of genus of such the above F. In general, the above procedure does not true since the polynomial F obtained from the formula (4.22) may be a reducible one whose genus does not exist, and this problem has been consulted in Lemma 4.2.2. Finally, we refer to Proposition 3.3.10 for an example of first-order AODEs of any positive genera which have no non-constant liouvillian solution.

4.3 Möbius transformations

A *Möbius transformation* is a transformation of the form

$$u = \frac{\alpha Y + \beta}{\gamma Y + \delta}, u' = \left(\frac{\alpha Y + \beta}{\gamma Y + \delta}\right)', \tag{4.34}$$

where $\alpha, \beta, \gamma, \delta \in \overline{\mathbb{C}(z)}$, $\alpha\delta - \beta\gamma \neq 0$. The inverse substitution of (4.34) is

$$Y = \frac{\delta u - \beta}{-\gamma u + \alpha}, Y' = \left(\frac{\delta u - \beta}{-\gamma u + \alpha}\right)'. \tag{4.35}$$

There is an expression (more details, see [32])

$$\begin{aligned}
\frac{\partial M(Y)}{\partial Y} &= \frac{\alpha\delta - \beta\gamma}{(\gamma Y + \delta)^2}, \\
\frac{\partial M(Y)}{\partial z} &= \frac{(\alpha'\gamma - \gamma'\alpha)Y^2 + (\alpha'\delta - \alpha\delta' + \beta'\gamma - \gamma'\beta)Y + \beta'\delta - \delta'\beta}{(\gamma Y + \delta)^2}, \\
u' = \frac{du}{dz} &= \frac{d(M(Y))}{dz} = \frac{\partial M(Y)}{\partial Y} Y' + \frac{\partial M(Y)}{\partial z}.
\end{aligned} \tag{4.36}$$

Definition 4.3.1. ([32, Definition 2.1]) Let $F(Y, Y') = \sum a_{ij} Y^i Y'^j$ be an irreducible polynomial over $\mathbb{C}(z)$ then we define the *differential total degree* of F by the number

$$\mu(F) = \max\{i + 2j \mid 0 \neq a_{ij} \in \mathbb{C}(z)\}.$$

By putting (4.34) into the AODE $G(u, u') = 0$ and using (4.36) we obtain

$$G(u, u') = G\left(\frac{\alpha Y + \beta}{\gamma Y + \delta}, \left(\frac{\alpha Y + \beta}{\gamma Y + \delta}\right)'\right) = \left(\frac{\alpha\delta - \beta\gamma}{\gamma Y + \delta}\right)^{\mu(G)} F(Y, Y') = 0. \tag{4.37}$$

In the reverse, from the formulas (4.35) and (4.37), we have

$$(-\gamma u + \alpha)^{\mu(F)} F\left(\frac{\delta u - \beta}{-\gamma u + \alpha}, \left(\frac{\delta u - \beta}{-\gamma u + \alpha}\right)'\right) = G(u, u') = 0. \tag{4.38}$$

Noting that $\mu(G) = \mu(F)$, see [32].

Definition 4.3.2. Let $F(Y, Y') = 0$ and $G(u, u') = 0$ be two first-order AODEs over $\mathbb{C}(z)$. We say F is *equivalent* to G if there is a Möbius transformation (4.34) such that the formula (4.38) is satisfied.

The Möbius transformation induces an equivalence relation of first-order AODEs, and it preserves the property of having an algebraic solution of the equivalence class. Since such Möbius transformations are birational then they also preserve the genus among the corresponding algebraic curves. Möbius transformations are well-studied in the works of [32, 39] for finding rational and algebraic solutions, therefore, there is no need to elaborate about them. In here, we prove that they also preserve the property of having a liouvillian solutions of the equivalence class.

Theorem 4.3.3. *[38, Theorem 3.1] Assume that F is equivalent to G. Then F has a liouvillian solution if and only if so does G. In the affirmative case, the correspondence of such solution is one to one.*

Proof. [32, Theorem 2.2] has shown that F and G obtain the same property of having an algebraic general solution. From formula (4.38), G has a liouvillian transcendental solution ξ if and only if

$$M^{-1}(\xi) = \frac{\delta\xi - \beta}{-\gamma\xi + \alpha}$$

is a transcendental solution F since

$$(-c\xi + a)^{\mu(F)} \neq 0.$$

By formula (4.34), the correspondence of solutions between F and G is one to one. □

Remark 4.3.4. Some examples of using the Möbius transformations for determining solutions of first-order or higher order AODEs can be found in [32, 39]. In particular, if we focus on algebraic solutions, in [32], such Möbius transformation is used to check if a certain AODE is belonged to an autonomous class of first-order AODEs. If this is the case, [1, Algorithm 4.4] can be applied to determine an algebraic general solution. From that, an algebraic solution of the original AODE can be returned.

In here, we show that Möbius transformation can be applied to the AODE (4.13) in Example 4.1.11. In fact, by substituting

$$Y = \frac{u-1}{z},$$

into the AODE (4.13) and using formula (4.38), we obtain

$$z^4 F\left(\frac{u-1}{z}, (\frac{u-1}{z})'\right) = G(u, u') = u'^2 - u^3 + u^2 = 0. \tag{4.39}$$

By Algorithm `LiouSolAut`, a liouvillian general solution of the AODE (4.39) is

$$(\exp i(z+c)+1)^2 u - 2\exp i(z+c) = 0, \quad i^2 = -1.$$

Therefore, a liouvillian general solution of the original AODE (4.13) is

$$(\exp i(z+c)+1)^2 (zY+1) - 2\exp i(z+c) = 0.$$

4.4 Liouvillian solutions of first-order AODEs with liouvillian coefficients

The content of this section is based on [38, Section 4]. Let us recall $\mathbb{C}(x)$ a differential field with the derivation $' = \dfrac{d}{dx}$ and let E be a liouvillian extension of $\mathbb{C}(x)$. Consider the differential equation

$$\tilde{F}(y, y') = 0, \tag{4.40}$$

where y is a function of x and $\tilde{F} \in E[y, w]$, i.e. first-order AODEs with the coefficients in a liouvillian extension E of $\mathbb{C}(x)$. For briefly, we call the equation (4.40) a first-order AODE with *liouvillian coefficients*. Our purpose is to transform the AODE (4.40) into the AODE (4.1) by means of change of variables $z = \varphi(x)$. Since Algorithm `LiouSol` is independent of the particular form of the indeterminate z, then z can be seen as a rational liouvillian element over $\mathbb{C}$ (see Definition 2.2.2). Therefore, Algorithm `LiouSol` can be extended to the case of solving first-order AODEs with liouvillian coefficients which can be converted into first-order AODEs over $\mathbb{C}(z)$ by a change of variable. Assume that there is a change

$$z = \varphi(x), \tag{4.41}$$

such that it turns an AODE (4.40) into (4.1), an AODE over $\mathbb{C}(z)$, i.e.

$$\tilde{F}(y, y') = F(Y, Y') = 0,$$

where $F \in \mathbb{C}(z)[y, w]$. If this occurs, then known-tools for finding liouvillian solutions of a first-order AODE can be applied. If $Y(z)$ is a liouvillian solution of (4.1), then $y(x) = Y \circ \varphi(x)$ is a liouvillan solution of (4.40).

Remark 4.4.1. In the spirit of symbolic computation, there are the same meaning between two differential fields $\left(\mathbb{C}(z), \dfrac{d}{dz}\right)$ and $\left(\mathbb{C}(x), \dfrac{d}{dx}\right)$. That means there are no difference between the two derivatives y' and Y' but

$$y' = \frac{dy}{dx}, \; Y' = \frac{dY}{dz}.$$

By the chain rule, a relation between y' and Y' is expressed

$$y' = \frac{dy}{dx} = \frac{d(Y \circ \varphi)}{dx} = \frac{dY}{d\varphi}\frac{d\varphi}{dx} = \frac{dY}{dz}\frac{dz}{dx} = Y'\frac{dz}{dx}.$$

The above expression may be applied to detect a candidate change of variables (4.41).

In the case of transcendental coefficients, we refer the readers to [4, Chapter V] for details. Here, we present some examples such that the change (4.41) can be applied.

Example 4.4.2. ([21, I·463, page 374]) Consider first-order AODE

$$yy'^2 - \exp(2x) = 0. \tag{4.42}$$

The coefficients of $\tilde{F}$ are in $\mathbb{C}(x)(\exp x)$. By setting $z = \varphi(x) = \exp x$, the given AODE is converted into an AODE over $\mathbb{C}(z)$

$$z^2(YY'^2 - 1) = 0.$$

After dividing z^2, we obtain an autonomous AODE ([21, I·462, page 373])

$$YY'^2 - 1 = 0, \tag{4.43}$$

which has a liouvillian general solution

$$Y = \sqrt[3]{\frac{9}{4}(z+c)^2}.$$

Therefore, a liouvillian general solution of the AODE (4.42) is

$$y = Y \circ \varphi = \sqrt[3]{\frac{9}{4}(\exp x + c)^2}.$$

Example 4.4.3. ([21, I·387, page 358]) Consider first-order AODE

$$y'^2 + (y' - y)\exp x = 0. \tag{4.44}$$

By setting $z = \varphi(x) = \exp x$, the AODE (4.44) is converted into an AODE

$$Y'^2 z^2 + Y' z^2 - Yz = 0. \tag{4.45}$$

The corresponding curve of (4.45) has a proper parametrization

$$\mathcal{P}(t) = (t^2 z + tz, t).$$

The associated ODE (see Algorithm `LiouSol` in Section 4.1.2) respect to $\mathcal{P}(t)$ is

$$t' = \frac{dt}{dz} = -\frac{t^2}{z(2t+1)}.$$

By symbolic integration (see [41]), the above ODE has only a general solution

$$\log(t^2 z) - \frac{1}{t} = c$$

which is not liouvillian solution, see [45]. Therefore, the AODE (4.45) has no liouvillian solution. That means the original AODE (4.44) has no liouvillian solutions.

In case of radical coefficients, assume that there is a change of variables

$$x = r(z) \in \mathbb{C}(z)$$

(by using [5, Algorithm 3.5]), then it always leads to the existence of the inverse substitution (4.41) $z = \varphi(x)$. Since z is algebraic over $\mathbb{C}(x)$ and

$$\frac{dz}{dx} = (\frac{dr}{dz})^{-1} \in \mathbb{C}(z),$$

then z is a rational liouvillian element over $\mathbb{C}$. In this case, Algorithm `LiouSol` can be applied to the case of solving first-order AODE with radical coefficients.

Example 4.4.4. [38, Example 4.4] Consider the first-order AODE

$$\tilde{F}(y, y') = -x\sqrt{x}y^3 + 4x^2y'^2 - 2xy^2 + 4xyy' - \sqrt{x}y + y^2 = 0 \qquad (4.46)$$

MAPLE 2022 finds a solution of the AODE (4.46) after hundreds of seconds and it is not an explicit form (i.e. it involves integral signs). On the other hand, by using [5, Algorithm 3.5], there is a change $z = \varphi(x) = \sqrt{x}$ which transforms AODE (4.46) into the AODE (4.13)

$$F(Y, Y') = -z^3Y^3 + z^2Y'^2 - 2z^2Y^2 + 2zYY' - zY + Y^2 = 0.$$

From Example 4.1.11, a liouvillian general solution of the AODE (4.46) is

$$(\exp i(\sqrt{x} + c) + 1)^2(\sqrt{x}y + 1) - 2\exp i(\sqrt{x} + c) = 0.$$

Remark 4.4.5. More examples of transforming first-order AODEs with radical coefficients into the AODEs (4.1) can be found in [5]. Note that, all of first-order AODEs obtained in here are of genus zero, hence, they are suitable for Algorithm `LiouSol`.

Conclusion

In this chapter, we first present Algorithm `LiouSol` for finding liouvillian solutions of first-order AODEs (4.1) of genus zero. In addition, we propose a method for solving a first-order AODE of positive genus via power transformations (Algorithm `RedPol` and Theorem 4.2.11). Finally, we consider the problem of solving first-order AODEs with coefficients in a liouvillian extension of $\mathbb{C}(x)$ by means of change of variables.

Conclusion and future work

We have considered the class of first-order AODEs and studied their liouvillian solutions. Several methods have been proposed to attack the problem of finding these solutions for a first-order AODE. In this dissertation, we have achieved the following main results.

1. We define a rational liouvillian solution (Definition 2.2.3) and give an algorithm (Algorithm `RatLiouSol` in Section 2.4) for finding rational liouvillian solutions of first-order autonomous AODEs.

2. We prove that liouvillian solutions (which include the class of algebraic solutions) of a first-order autonomous AODE of genus zero must be rational liouvillian solutions (Lemma 3.2.2) and propose an algorithm (Algorithm `LiouSolAut` in Section 3.3) for finding and classifying such a liouvillian solution in algebraic and transcendental cases.

3. We give an algorithm (Algorithm `LiouSol` in Section 4.1.2) for finding liouvillian solutions of first-order AODEs of genus zero (included autonomous and non-autonomous cases).

4. We define power transformations (Definition 4.2.1) and propose an algorithm (Algorithm `RedPol` in Section 4.2.2) to obtain the reduced form of a first-order AODE. This result leads to a method for finding liouvillian solutions of certain first-order AODEs of positive genera in the case that their reduced forms are of genus zero (Section 4.2.3).

5. We transform the problem of solving first-order AODEs with liouvillian coefficients into the case of solving an AODE (4.1) by means of change of variables (Section 4.4).

The following is a short description of our future research.

1. **Study on the relation of the positive genera of first-order AODEs which are generated by substituting a power transformation** (4.19) **into the ones of genus zero.** In Section 4.2, we have considered this problem but not yet to give an explicit relations of such genera. To attack this problem, we are working on related documents [8, 24, 27, 31].

2. **Keep focusing on the problem of determining liouvillian solutions of first-order ODEs** (4.7). This problem has been consulted in Section 4.1.3 and we will keep it going by focusing on the related works [4, 9, 53–55].

List of author's related publications

1. Nguyen T. D., Ngo L. X. C. (2021), "Rational liouvillian solutions of algebraic ordinary differential equations of order one", *Acta Mathematica Vietnamica*, 46 (4), pp. 689–700.

2. Nguyen T. D., Ngo L. X. C. (2023), "Liouvillian solutions of algebraic ordinary differential equations of order one of genus zero", *Journal of Systems Science and Complexity*, 36(2), pp. 884–893.

3. Nguyen T. D., Ngo L. X. C., "Liouvillian solutions of first-order algebraic ordinary differential equations", submitted.

4. Nguyen T. D. (2024), "Finding liouvillian solutions of first-order algebraic ordinary differential equations by change of variables", *Quy Nhon University Journal of Science*, 18(3), pp. 83–89.

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Curriculum vitae

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Publications and preprints

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- Nguyen T. D., Ngo L. X. C. (2021), "Rational liouvillian solutions of algebraic ordinary differential equations of order one", *Acta Mathematica Vietnamica*, 46 (4), pp. 689–700.
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- Nguyen T. D., Ngo L. X. C., "Liouvillian solutions of first-order algebraic ordinary differential equations", submitted.
- Nguyen T. D. (2024), "Finding liouvillian solutions of first-order algebraic ordinary differential equations by change of variables", *Quy Nhon University Journal of Science*, 18(3), pp. 83–89.

Conferences and talks

- International Workshop on Matrix Analysis and Its Applications, July 7–8, 2023, Quy Nhon, Viet Nam.
- 10th Viet Nam Mathematical Congress, August 8–12, 2023, Da Nang, Viet Nam.
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- Resonances in the Mathematical World, August 1–4, 2024, Ho Chi Minh City, Viet Nam.